

Interest Rates and Time Preferences in Continuous-Time Asset Markets

Daniel Graydon Stephenson

Abstract

This paper investigates asset markets where transactions occur continuously over time. Such markets are shown to possess a unique Nash equilibrium price path. If the interest rate is aligned with the subjective discount rate, rationality is shown to be a sufficient condition for the equilibrium price path. Conversely, if the interest rate is misaligned with the subjective discount rate, then deviations from the equilibrium price path are shown to be consistent with rationality. These results suggest that misalignment between interest rates and time preferences may contribute to asset price instability.

1 Continuous Asset Markets

Consider a market for uniform assets populated by a continuum of agents $i \in [0, 1]$. Let $q_i(t) \in \mathbb{R}$ denote the quantity of assets held by agent i at time t . Let $p(t) \in \mathbb{R}_+$ denote the asset price at time $t \in \mathbb{R}_+$. Let $m_i(t) \in \mathbb{R}$ denote the quantity of money held by agent i at time t . Agent i 's total wealth at time t is given by

$$w_i(t) = m_i(t) + p(t) q_i(t) \quad (1)$$

Here $p(t) q_i(t)$ denotes the value of the assets held by agent i at time t . Let $r \in \mathbb{R}_{++}$ denote the interest rate and let $z \in \mathbb{R}_{++}$ denote the rental rate on assets. Let $\dot{g}(t)$ denote the right derivative of g with respect to t for all $g : \mathbb{R}_+ \rightarrow \mathbb{R}$. Agent i 's net revenue at time t is given by

$$\dot{m}_i(t) = r m_i(t) + z q_i(t) - p(t) \dot{q}_i(t) \quad (2)$$

Here $r m_i(t)$ denotes interest income, $z q_i(t)$ denotes rental income, and $p(t) \dot{q}_i(t)$ denotes the cost of asset purchases. The net rate of change in agent i 's wealth at time t is given by

$$\dot{w}_i(t) = r m_i(t) + z q_i(t) + \dot{p}(t) q_i(t) \quad (3)$$

Here $\dot{p}(t) q_i(t)$ denotes agent i 's rate of capital gains at time t . Let $\bar{p} \in \mathbb{R}$ denote the present value of the future rental cash flows generated by an asset such that

$$\bar{p} = \int_0^{\infty} e^{-rt} z dt = \frac{z}{r} \quad (4)$$

Let $\zeta(t|p)$ denote the economic rent on an asset at time t such that

$$\zeta(t|p) = z + \dot{p}(t) - r p(t) \quad (5)$$

The economic rent on an asset is equal to the accounting rent z plus the appreciation rate $\dot{p}(t)$ less the opportunity cost $r p(t)$. Let $\beta \geq r$ denote the

subjective discount rate. Let $\lambda(t|p)$ denote the discounted future economic rent such that

$$\lambda(t|p) = \int_t^\infty e^{-\beta(s-t)} \zeta(s|p) ds \quad (6)$$

Let π_i denote agent i 's total discounted economic profit such that

$$\pi_i = \int_0^\infty e^{-\beta s} \zeta(t|p) q_i(t) dt \quad (7)$$

Let T denote an exponentially distributed stochastic time such that $\Pr(T \leq t) = 1 - e^{-\beta t}$. Proposition 1 states that agent i 's expected wealth at time T is an affine function of her total discounted economic profit.

Proposition 1. *The expected value of agent i 's wealth at time T is given by*

$$E\{w_i(T)\} = \frac{\beta}{\beta - r} \left[w_i(0) + \pi_i \right] \quad \text{for } \beta > r \quad (8)$$

2 Investment Strategies

Let P denote the set of continuous and right-differentiable price paths $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Let $p_t : [0, t] \rightarrow \mathbb{R}_+$ denote the restriction of the price path $p \in P$ to the closed interval $[0, t]$. Let \mathcal{H} denote the set of all possible price histories such that

$$\mathcal{H} = \{p_t : p \in P, t \in \mathbb{R}_+\} \quad (9)$$

Let $\mathcal{D} = [-\gamma, \gamma]$ denote the set of feasible purchase rates where $\gamma \in \mathbb{R}_{++}$. Agent i 's investment strategy $\phi_i : \mathcal{H} \rightarrow \mathcal{D}$ specifies her net purchase rate as a function of the price history such that

$$q_i(t|\phi_i, p) = q_i(0) + \int_0^t \phi_i(p_s) ds$$

Let $\pi_i(\phi_i|p)$ denote agent i 's total discounted economic profit as a function of her investment strategy under the price path p such that

$$\pi_i(\phi_i|p) = \int_0^\infty e^{-\beta s} \zeta(t|p) q_i(t|\phi_i, p) dt \quad (10)$$

Let Φ_i denote the set of investment strategies $\phi_i : \mathcal{H} \rightarrow \mathcal{D}$ such that $\dot{q}_i(t|\phi_i, p)$ is right-continuous for all $p \in P$. Let $\Phi = \prod_{i \in [0,1]} \Phi_i$ denote the set of investment strategy profiles. An investment strategy $\phi_i \in \Phi_i$ is said to be optimal under the price path $p \in P$ if it maximizes $\pi_i(\phi_i|p)$ over Φ_i . Proposition 2 characterizes the set of optimal investment strategies. It states that optimal investment strategies involve purchasing assets when the discounted future economic rent is positive and selling assets when the discounted future economic rent is negative.

Proposition 2. *An investment strategy $\phi_i \in \Phi_i$ is optimal under the price path $p \in P$ if and only if*

$$\dot{q}_i(t|\phi_i, p) |\operatorname{sgn}(\lambda(t|p))| = \gamma \operatorname{sgn}(\lambda(t|p)) \quad \text{for all } t \in \mathbb{R}_+$$

3 Supply and Demand

Let $Q(t)$ denote the total quantity of assets demanded by investors at time t such that

$$Q(t) = \int_0^1 q_i(t) di \quad (11)$$

The asset price is given by the inverse supply function $f : \mathbb{R} \rightarrow \mathbb{R}_{++}$ such that $p(t) = f(Q(t))$. Here f is assumed to be an increasing continuously differentiable bijection with bounded elasticity. Let p_ϕ denote the price path generated by ϕ such that

$$p_\phi(t) = f(Q_\phi(t)) \quad (12)$$

$$Q_\phi(t) = \int_0^1 q_i(t|\phi_i, p_\phi) di \quad (13)$$

4 Nash Equilibrium

An investment strategy $\phi_i \in \Phi_i$ is a best response to the opponent strategy profile $\phi_{-i} \in \Phi_{-i}$ if it maximizes agent i 's payoff under the price path p_ϕ generated by ϕ . Let $\Phi_i^* : \Phi_{-i} \rightrightarrows \Phi_i$ denote agent i 's best response correspondence such that

$$\Phi_i^*(\phi_{-i}) = \operatorname{argmax}_{\phi_i \in \Phi_i} \pi_i(\phi_i | p_\phi) \quad (14)$$

A strategy profile $\phi \in \Phi_f$ is a Nash equilibrium if $\phi_i \in \Phi_i^*(\phi_{-i})$ for all $i \in [0, 1]$. Proposition 3 establishes the existence of a Nash equilibrium investment strategy profile.

Proposition 3. *There exists a Nash equilibrium $\phi^* \in \Phi$ such that*

$$\phi_i^*(p_t) = \gamma \operatorname{sgn}(\bar{p} - p(t)) \quad (15)$$

A price path $p \in P$ is said to be an equilibrium price path if it is generated by a Nash equilibrium ϕ such that $p = p_\phi$. Proposition 4 characterizes the unique Nash equilibrium price path.

Proposition 4. *The unique Nash equilibrium price path p^* is given by*

$$p^*(t) = \begin{cases} f(Q(0) + \operatorname{sgn}(\bar{p} - p(0)) \gamma t) & \text{if } t < t^* \\ \bar{p} & \text{if } t \geq t^* \end{cases} \quad (16)$$

$$\text{where } t^* = \gamma^{-1} |f^{-1}(\bar{p}) - f^{-1}(p(0))|$$

The equilibrium price path is completely determined by the fundamental value \bar{p} , the transaction rate γ , and the initial price $p(0)$. In contrast, the equilibrium price path is independent of the subjective discount rate β .

5 Rational Price Paths

A price path $p \in P$ is said to be feasible if it is generated by an investment strategy profile $\phi \in \Phi$. Let P_f denote the set of feasible price paths such that

$$P_f = \{p_\phi : \phi \in \Phi\} \quad (17)$$

A price history $p_t \in \mathcal{H}$ is said to be feasible if it is the restriction of a feasible price path. Let \mathcal{H}_f denote the set of feasible price histories such that

$$\mathcal{H}_f = \{p_t : p \in P_f, t \in \mathbb{R}_+\} \quad (18)$$

Let p_t^+ denote the set of feasible price paths consistent with the price history $p_t \in \mathcal{H}_f$ such that

$$p_t^+ = \{\rho \in P_f : \rho_t = p_t\} \quad (19)$$

Let μ_i denote agent i 's beliefs regarding the price path conditional on the observed price history such that agent i 's subjective probability of the event $p \in E$ conditional the feasible price history $p_t \in \mathcal{H}_f$ is given by $\mu_i(E|p_t)$ for all $E \in \mathcal{A}(p_t)$ where $\mathcal{A}(p_t)$ is a sigma algebra over p_t^+ . Let M_i denote the set of beliefs μ_i such that $\mu_i(p_t^+|p_t) = 1$ for all $p_t \in \mathcal{H}_f$. Agent i 's subjective expected payoff conditional on the observing the feasible price history $p_t \in \mathcal{H}_f$ is given by

$$\bar{\pi}_i(\phi_i|\mu_i, p_t) = \int_{p_t^+} \pi_i(\phi_i|\rho) d\mu_i(\rho|p_t)$$

An investment strategy $\phi_i \in \Phi_i$ is said to be rational if there exists a belief $\mu_i \in M_i$ rationalizing ϕ_i such that

$$\phi_i \in \operatorname{argmax}_{\varphi_i \in \Phi_i} \bar{\pi}_i(\varphi_i|\mu_i, p_t) \quad \text{for all } p_t \in \mathcal{H}_f \quad (20)$$

Let Φ_0 denote the set of rational investment strategy profiles $\phi \in \Phi$ such that ϕ_i is rational for all $i \in [0, 1]$. A price path $p \in P$ is said to be rational if

it is generated by a rational investment strategy profile such that $p = p_\phi$ for some $\phi \in \Phi_0$. Let P_0 denote the set of rational price paths. Proposition 5 states that if the interest rate equals the subjective discount rate then the equilibrium price path is the only rational price path.

Proposition 5. *If $r = \beta$ then $P_0 = \{p^*\}$.*

Proposition 6 states that some rational price paths diverge from the equilibrium price path if the interest rate is below the subjective discount rate.

Proposition 6. *If $r < \beta$ then there exists $p \in P_0$ such that $p \neq p^*$.*

These results indicate that rationality is only sufficient for the equilibrium price path when the interest rate is aligned with the subjective discount rate.

A Proofs

Proof of Proposition 1. Since $w_i(t) = m_i(t) + p(t) q_i(t)$ we have

$$\begin{aligned}
 \dot{w}_i(t) &= r m_i(t) + z q_i(t) + \dot{p}(t) q_i(t) \\
 &= r [w_i(t) - p(t) q_i(t)] + z q_i(t) + \dot{p}(t) q_i(t) \\
 &= r w_i(t) + [z + \dot{p}(t) - r p(t)] q_i(t) \\
 &= r w_i(t) + \zeta(t|p) q_i(t) \\
 w_i(t) &= e^{rt} w_i(0) + e^{rt} \int_0^t e^{-r\tau} \zeta(\tau|p) q_i(\tau) d\tau
 \end{aligned}$$

If $\beta > r$ then the expected value of $w_i(T)$ is given by

$$\begin{aligned}
 E \{w_i(T)\} &= E \{e^{rT} w_i(0)\} + E \left\{ e^{rT} \int_0^T e^{-rs} \zeta(s|p) q_i(s) ds \right\} \\
 &= \frac{\beta w_i(0)}{\beta - r} + \beta \int_0^\infty e^{(r-\beta)t} \int_0^t e^{-rs} \zeta(s|p) q_i(s) ds dt
 \end{aligned}$$

The second term can be written as

$$\begin{aligned}
 &\beta \int_{s=0}^{s=\infty} e^{-rs} \zeta(s|p) q_i(s) \int_{t=s}^{t=\infty} e^{(r-\beta)t} dt ds \\
 &= \beta \int_{s=0}^{s=\infty} e^{-rs} \zeta(s|p) q_i(s) \left[\frac{e^{(r-\beta)s}}{\beta - r} \right] ds \\
 &= \frac{\beta}{\beta - r} \int_0^\infty e^{-\beta s} \zeta(s|p) q_i(s) ds = \frac{\beta \pi_i}{\beta - r}
 \end{aligned}$$

Hence the expected value of $w_i(T)$ is given by

$$E \{w_i(T)\} = \frac{\beta}{\beta - r} \left[w_i(0) + \pi_i \right]$$

□

Lemma 1. $\pi_i(\phi_i|p) = q_i(0)\lambda(0|p) + u_i(\phi_i|p)$ where

$$u_i(\phi_i|p) = \int_0^\infty e^{-\beta t} \dot{q}_i(t|\phi_i, p) \lambda(t|p) dt$$

Proof. Agent i 's total discounted economic profit is given by

$$\begin{aligned} \pi_i(\phi_i|p) &= \int_0^\infty e^{-\beta s} \zeta(s|p) q_i(s) ds \\ &= \int_0^\infty e^{-\beta s} \zeta(s|p) \left[q_i(0) + \int_0^s \dot{q}_i(t|\phi_i, p) dt \right] ds \\ &= q_i(0)\lambda(0|p) + \int_0^\infty e^{-\beta s} \zeta(s|p) \int_0^s \dot{q}_i(t|\phi_i, p) dt ds \end{aligned}$$

The second term can be written as

$$\begin{aligned} &\int_{t=0}^{t=\infty} \dot{q}_i(t|\phi_i, p) \int_{s=t}^{s=\infty} e^{-\beta s} \zeta(s|p) ds dt \\ &= \int_{t=0}^{t=\infty} e^{-\beta t} \dot{q}_i(t|\phi_i, p) e^{\beta t} \int_{s=t}^{s=\infty} e^{-\beta s} \zeta(s|p) ds dt \\ &= \int_0^\infty e^{-\beta t} \dot{q}_i(t|\phi_i, p) \lambda(t|p) dt \end{aligned}$$

□

Proof of Proposition 2. If $\dot{q}_i(t|\phi_i, p) = \gamma$ for $\lambda(t|p) > 0$ and $\dot{q}_i(t|\phi_i, p) = -\gamma$ for $\lambda(t|p) < 0$ then ϕ_i maximizes $\pi_i(\phi_i|p)$ over Φ_i by Lemma 1. Conversely, suppose $\dot{q}_i(t|\phi_i, p) \neq \gamma \operatorname{sgn}(\lambda(t|p))$ for some $t \in \mathbb{R}_+$ where $\lambda(t|p) \neq 0$. Since $\lambda(t|p)$ is continuous and $\dot{q}_i(t|\phi_i, p)$ is right-continuous, there exists $\varepsilon > 0$ such that $\lambda(s|p) \neq 0$ and $\dot{q}_i(s|\phi_i, p) \neq \gamma \operatorname{sgn}(\lambda(t|p))$ for all $s \in (t, t + \varepsilon)$. But then ϕ_i is suboptimal by Lemma 1. □

Proof of Proposition 3. Let $\phi \in \Phi$ be an investment strategy profile such that

$$\phi_i(p_t) = \gamma \operatorname{sgn}(\bar{p} - p(t))$$

Let $t^* \in \mathbb{R}_+$ denote the time at which p_ϕ reaches \bar{p} such that

$$\gamma t^* = |f^{-1}(\bar{p}) - f^{-1}(p(0))|$$

Hence the price path p_ϕ satisfies

$$\begin{aligned} p_\phi(t) &= \bar{p} = \frac{z}{r} \quad \text{if } t \geq t^* \\ \operatorname{sgn}(\dot{p}_\phi(t)) &= \operatorname{sgn}(\bar{p} - p(0)) \quad \text{if } t < t^* \end{aligned}$$

By (6) we have

$$\begin{aligned} \lambda(t|p_\phi) &= \int_t^\infty e^{-\beta(s-t)} \zeta(s|p_\phi) ds \\ \lambda(t|p_\phi) &= \int_t^\infty e^{-\beta(s-t)} [z + \dot{p}_\phi(t) - rp_\phi(t)] ds \\ \lambda(t|p_\phi) &= \int_t^\infty e^{-\beta(s-t)} [z + 0 - z] ds = 0 \quad \text{if } t \geq t^* \\ \operatorname{sgn}(\lambda(t|p)) &= \operatorname{sgn}(\bar{p} - p(0)) \quad \text{if } t < t^* \end{aligned}$$

So ϕ_i is optimal under p_ϕ by Proposition 2. □

Lemma 2. *The expected value of $\dot{p}(T)$ conditional on $T \geq t$ is given by*

$$E_t \{\dot{p}(T)\} = \beta E_t \{p(T)\} - \beta p(t)$$

Proof. Since $T \sim \text{Exp}(\beta)$ the expected price $p(T)$ at time T satisfies

$$\begin{aligned} E_t \{p(T)\} &= p(t) + E_t \left\{ \int_t^T \dot{p}(x) dx \right\} \\ E_t \{p(T)\} &= p(t) + \beta e^{\beta t} \int_{s=t}^{s=\infty} e^{-\beta s} \int_{x=t}^{x=s} \dot{p}(x) dx ds \\ E_t \{p(T)\} &= p(t) + \beta e^{\beta t} \int_{x=t}^{x=\infty} \dot{p}(x) \int_{s=x}^{s=\infty} e^{-\beta s} ds dx \\ E_t \{p(T)\} &= p(t) + \beta e^{\beta t} \int_{x=t}^{x=\infty} \dot{p}(x) \left[\frac{e^{-\beta x}}{\beta} \right] dx \\ E_t \{p(T)\} &= p(t) + e^{\beta t} \int_{x=t}^{x=\infty} e^{-\beta x} \dot{p}(x) dx \\ E_t \{p(T)\} &= p(t) + \frac{1}{\beta} E_t \{\dot{p}(T)\} \\ E_t \{\dot{p}(T)\} &= \beta E_t \{p(T)\} - \beta p(t) \end{aligned}$$

□

Lemma 3. $\lambda(t|p) = \alpha\bar{p} + (1 - \alpha) E_t \{p(T)\} - p(t)$ where $\alpha = \frac{r}{\beta}$

Proof. By Lemma 2 we have

$$E_t \{\dot{p}(T)\} = \beta E_t \{p(T)\} - \beta p(t)$$

By (6) the future discounted economic rent is given by

$$\begin{aligned} \lambda(t|p) &= \int_t^\infty e^{-\beta(s-t)} \zeta(s|p_\phi) ds \\ \beta\lambda(t|p) &= \beta \int_t^\infty e^{-\beta(s-t)} \zeta(s|p_\phi) ds = E_t \{\zeta(T|p)\} \\ \beta\lambda(t|p) &= E_t \{z + \dot{p}(T) - rp(T)\} \\ \beta\lambda(t|p) &= z + E_t \{\dot{p}(T)\} - rE_t \{p(T)\} \\ \beta\lambda(t|p) &= z + \beta E_t \{p(T)\} - \beta p(t) - rE_t \{p(T)\} \\ \beta\lambda(t|p) &= z - \beta p(t) + (\beta - r) E_t \{p(T)\} \\ \lambda(t|p) &= \frac{z}{\beta} - p(t) + \left(\frac{\beta - r}{\beta}\right) E_t \{p(T)\} \\ \lambda(t|p) &= \left(\frac{r}{\beta}\right) \frac{z}{r} - p(t) + \left(\frac{\beta - r}{\beta}\right) E_t \{p(T)\} \\ \lambda(t|p) &= \alpha\bar{p} - p(t) + (1 - \alpha) E_t \{p(T)\} \end{aligned}$$

□

Lemma 4. For every feasible price path $p \in P_f$ we have

$$E_t \{p(T)\} \leq \int_0^\infty e^{-s} f \left(f^{-1}(p(t)) + \frac{\gamma}{\beta} s \right) ds$$

Proof. If $p \in P_f$ then $p(t) = f(Q_\phi(t))$ for some $\phi \in \Phi$ so

$$p(t+s) \leq f \left(f^{-1}(p(t)) + \gamma s \right)$$

Taking expectation conditional on $T \geq t$ yields

$$\begin{aligned} E_t \{p(T)\} &= \beta \int_0^\infty e^{-\beta s} p(t+s) ds \\ &= \beta \int_0^\infty e^{-\beta s} f \left(f^{-1}(p(t)) + \gamma s \right) ds \\ &= \int_0^\infty e^{-u} f \left(f^{-1}(p(t)) + \frac{\gamma}{\beta} u \right) du \quad \text{where } u = \beta s \end{aligned}$$

□

Lemma 5. There exists $\eta \in \mathbb{R}_{++}$ such that for all $a, b \in \mathbb{R}_{++}$

$$\frac{f(b)}{f(a)} < \left(\frac{b}{a} \right)^\eta$$

Proof. Since f is continuously differentiable with bounded elasticity there exists $\eta \in \mathbb{R}_{++}$ such that

$$\begin{aligned} f'(x) \frac{x}{f(x)} &< \eta \quad \text{for all } x > 0 \\ \frac{f'(x)}{f(x)} &< \frac{\eta}{x} \\ \int_a^b \frac{f'(x)}{f(x)} dx &< \int_a^b \frac{\eta}{x} dx \\ \log(f(b)) - \log(f(a)) &< \eta \log(b) - \eta \log(a) \\ \frac{f(b)}{f(a)} &< \left(\frac{b}{a} \right)^\eta \end{aligned}$$

□

Lemma 6. $\lim_{x \rightarrow \infty} \int_0^\infty e^{-s} \frac{f(f^{-1}(x) + \gamma s)}{x} ds = 1$

Proof. Since f is increasing and bijective $\lim_{x \rightarrow \infty} f^{-1}(x) = \infty$ and

$$\begin{aligned} \frac{f(f^{-1}(x) + \gamma s)}{x} &\geq 1 \quad \text{for } s \geq 0 \\ \int_0^\infty e^{-s} \frac{f(f^{-1}(x) + \gamma s)}{x} ds &\geq 1 \end{aligned}$$

By Lemma 5 we have

$$\begin{aligned} \frac{f(f^{-1}(x) + \gamma s)}{x} &= \frac{f(f^{-1}(x) + \gamma s)}{f(f^{-1}(x))} \\ \frac{f(f^{-1}(x) + \gamma s)}{x} &< \left(\frac{f^{-1}(x) + \gamma s}{f^{-1}(x)} \right)^\eta \quad \text{for } x > f(0) \\ \frac{f(f^{-1}(x) + \gamma s)}{x} &< \left(1 + \frac{\gamma s}{f^{-1}(x)} \right)^\eta \\ \int_0^\infty e^{-s} \frac{f(f^{-1}(x) + \gamma s)}{x} ds &< \int_0^\infty e^{-s} \left(1 + \frac{\gamma s}{f^{-1}(x)} \right)^\eta ds \end{aligned}$$

Since $\lim_{x \rightarrow \infty} f^{-1}(x) = \infty$ taking the limit as $x \rightarrow \infty$ obtains

$$\lim_{x \rightarrow \infty} \int_0^\infty e^{-s} \left(1 + \frac{\gamma s}{f^{-1}(x)} \right)^\eta ds = 1$$

□

Lemma 7. *There exists $\hat{p} > \bar{p}$ such that*

$$p(t) > \hat{p} \implies \lambda(t|p) < 0 \quad \text{for all } p \in P_f$$

Proof. By Lemma 3 we have

$$\lambda(t|p) = \alpha\bar{p} + (1 - \alpha) E_t \{p(T)\} - p(t)$$

By Lemma 4 we can write

$$\lambda(t|p) \leq \alpha\bar{p} + (1 - \alpha) \int_0^\infty e^{-t} f \left(f^{-1}(p(t)) + \frac{\gamma}{\beta} t \right) dt - p(t)$$

Dividing by $p(t)$ obtains

$$\frac{\lambda(t|p)}{p(t)} \leq \alpha \frac{\bar{p}}{p(t)} + (1 - \alpha) \int_0^\infty e^{-t} \frac{f \left(f^{-1}(p(t)) + \frac{\gamma}{\beta} t \right)}{p(t)} dt - 1$$

By Lemma 6 taking the limit as $p(t) \rightarrow \infty$ obtains

$$\lim_{x \rightarrow \infty} \alpha \frac{\bar{p}}{p(t)} + (1 - \alpha) \int_0^\infty e^{-t} \frac{f \left(f^{-1}(p(t)) + \frac{\gamma}{\beta} t \right)}{p(t)} dt - 1 = -\alpha$$

□

Lemma 8. *There exists $\check{p} \in (0, \bar{p})$ such that*

$$p(t) < \check{p} \implies \lambda(t|p) > 0 \quad \text{for all } p \in P_f$$

Proof. By Lemma 3 we have

$$\lambda(t|p) = \alpha\bar{p} + (1 - \alpha) E_t \{p(T)\} - p(t)$$

Since $p : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$ and $\alpha = \frac{\tau}{\beta} \in (0, 1)$

$$\lambda(t|p) \geq \alpha\bar{p} - p(t)$$

Hence $\lambda(t|p) > 0$ if $p(t) < \alpha\bar{p}$.

□

Lemma 9. *If $\phi \in \Phi$ is a Nash equilibrium and $p_\phi(t) > \bar{p}$ then $\dot{Q}_\phi(t) = -\gamma$.*

Proof. By Lemma 7 there exists $\hat{p}_0 > \bar{p}$ such that for all $p \in P_f$

$$p(t) > \hat{p}_0 \implies \lambda(t|p) < 0$$

For all $n \in \mathbb{N}$ let $\hat{p}_n \in \mathbb{R}_{++}$ such that

$$\hat{p}_n = \frac{z}{\beta} \sum_{k=0}^{n-1} \left(\frac{\beta - r}{\beta} \right)^k + \left(\frac{\beta - r}{\beta} \right)^n \hat{p}_0$$

Then by Proposition 2

$$p_\phi(t) > \hat{p}_0 \implies \dot{Q}_\phi(t) = -\gamma$$

For the inductive hypothesis, suppose

$$p_\phi(t) > \hat{p}_n \implies \dot{Q}_\phi(t) = -\gamma$$

Hence if $p_\phi(t) \leq \hat{p}_n$ then $E_t \{p_\phi(T)\} \leq \hat{p}_n$ and by Lemma 3

$$\begin{aligned} \lambda(t|p_\phi) &\leq z - \beta p_\phi(t) + (\beta - r) \hat{p}_n \\ \lambda(t|p_\phi) &\leq \beta [\hat{p}_{n+1} - p_\phi(t)] \end{aligned}$$

Then by Proposition 2

$$p_\phi(t) > \hat{p}_{n+1} \implies \dot{Q}_\phi(t) = -\gamma$$

So by induction we have

$$p_\phi(t) > \hat{p}_n \implies \dot{Q}_\phi(t) = -\gamma \quad \text{for all } n \in \mathbb{N}$$

Taking the limit as $n \rightarrow \infty$ obtains

$$\lim_{n \rightarrow \infty} \hat{p}_n = \frac{z}{\beta} \sum_{k=0}^{\infty} \left(\frac{\beta - r}{\beta} \right)^k = \frac{z}{\beta} \left(\frac{\beta}{r} \right) = \frac{z}{r} = \bar{p}$$

□

Lemma 10. *If $\phi \in \Phi$ is a Nash equilibrium and $p_\phi(t) < \bar{p}$ then $\dot{Q}_\phi(t) = \gamma$.*

Proof. By Lemma 8 there exists $\check{p}_0 < \bar{p}$ such that for all $p \in P_f$

$$p(t) < \check{p}_0 \implies \lambda(t|p) > 0$$

For all $n \in \mathbb{N}$ let $\check{p}_n \in \mathbb{R}_{++}$ such that

$$\check{p}_n = \frac{z}{\beta} \sum_{k=0}^{n-1} \left(\frac{\beta - r}{\beta} \right)^k + \left(\frac{\beta - r}{\beta} \right)^n \check{p}_0$$

Then by Proposition 2

$$p_\phi(t) < \check{p}_0 \implies \dot{Q}_\phi(t) = \gamma$$

For the inductive hypothesis, suppose

$$p_\phi(t) < \check{p}_n \implies \dot{Q}_\phi(t) = \gamma$$

Hence if $p_\phi(t) \geq \check{p}_n$ then $E_t \{p_\phi(T)\} \geq \check{p}_n$ and by Lemma 3

$$\begin{aligned} \lambda(t|p_\phi) &\geq z - \beta p_\phi(t) + (\beta - r) \check{p}_n \\ \lambda(t|p_\phi) &\geq \beta [\check{p}_{n+1} - p_\phi(t)] \end{aligned}$$

Then by Proposition 2

$$p_\phi(t) < \check{p}_{n+1} \implies \dot{Q}_\phi(t) = \gamma$$

So by induction we have

$$p_\phi(t) < \check{p}_n \implies \dot{Q}_\phi(t) = \gamma \quad \text{for all } n \in \mathbb{N}$$

Taking the limit as $n \rightarrow \infty$ obtains

$$\lim_{n \rightarrow \infty} \check{p}_n = \frac{z}{\beta} \sum_{k=0}^{\infty} \left(\frac{\beta - r}{\beta} \right)^k = \frac{z}{\beta} \left(\frac{\beta}{r} \right) = \frac{z}{r} = \bar{p}$$

□

Lemma 11. *If $\phi \in \Phi$ is a Nash equilibrium and $p_\phi(t) = \bar{p}$ then $\dot{Q}_\phi(t) = 0$.*

Proof. By (12) the price path generated by ϕ satisfies

$$\dot{p}_\phi(s) = f'(Q_\phi(s)) \dot{Q}_\phi(s)$$

Now $\dot{Q}_\phi(t)$ is right-continuous in t since $q_i(t) = \phi_i(p_t)$ is right-continuous in t . Hence, if $\dot{Q}_\phi(t) > 0$ then there exists $s > t$ such that $\dot{Q}_\phi(\ell) > 0$ for $\ell \in [t, s]$. But then $p_\phi(s) > \bar{p}$ and $\dot{p}_\phi(s) > 0$, which contradicts Lemma 9. Conversely, if $\dot{Q}_\phi(t) < 0$ then there exists $s > t$ such that $p_\phi(s) < \bar{p}$ and $\dot{p}_\phi(s) < 0$, which contradicts Lemma 10. \square

Proof of Proposition 4. By Proposition 3 there exists a Nash equilibrium investment strategy profile ϕ^* such that $p_{\phi^*}(t) = p^*(t)$ for all $t \in \mathbb{R}_+$. Suppose ϕ is a Nash equilibrium strategy profile. By Lemma 9, $\dot{Q}_\phi(t) = -\gamma$ if $p_\phi(t) > \bar{p}$. By Lemma 10, $\dot{Q}_\phi(t) = \gamma$ if $p_\phi(t) < \bar{p}$. Let $t^* = \gamma^{-1} |f^{-1}(\bar{p}) - Q_\phi(0)|$. Since $p_\phi(t) = f(Q_\phi(t))$ we have $Q_\phi(t) = Q_\phi(0) + \text{sgn}(\bar{p} - p_\phi(0)) \gamma t$ for all $t < t^*$ and $Q_\phi(t^*) = f^{-1}(\bar{p})$. By Lemma 11, $\dot{Q}_\phi(t) = 0$ if $p_\phi(t) = \bar{p}$. Hence $Q_\phi(t) = f^{-1}(\bar{p})$ for all $t \geq t^*$. \square

Proof of Proposition 5. If $r = \beta$ then by Lemma 3 the total discounted economic rent is given by $\lambda(t|p) = \bar{p} - p(t)$. By Lemma 1, agent i 's payoff under the price path $p \in P$ can be written as

$$\begin{aligned}\pi_i(\phi_i|p) &= q_i(0)\lambda(0|p) + \int_0^\infty e^{-\beta t} \dot{q}_i(t|\phi_i, p) \lambda(t|p) dt \\ &= q_i(0)[\bar{p} - p(0)] + \int_0^\infty e^{-\beta t} \dot{q}_i(t|\phi_i, p) [\bar{p} - p(t)] dt\end{aligned}$$

If ϕ is a rational strategy profile then ϕ_i is a rational investment strategy and there exists $\mu_i \in M_i$ such that

$$\begin{aligned}\phi_i &\in \operatorname{argmax}_{\varphi_i \in \Phi_i} \int_{p_t^+} \pi_i(\varphi_i|\rho) d\mu_i(\rho|p_t) \quad \text{for all } p_t \in \mathcal{H}_f \\ \phi_i &\in \operatorname{argmax}_{\varphi_i \in \Phi_i} \int_{p_t^+} \int_0^\infty e^{-\beta t} \dot{q}_i(t|\varphi_i, \rho) [\bar{p} - \rho(t)] dt d\mu_i(\rho|p_t)\end{aligned}$$

Hence for all ρ in the support of $\mu_i(\cdot|p_t)$ we have

$$\begin{aligned}\dot{q}_i(t|\phi_i, \rho_t) &= \gamma \quad \text{if } \rho(t) < \bar{p} \\ \dot{q}_i(t|\phi_i, \rho_t) &= -\gamma \quad \text{if } \rho(t) > \bar{p}\end{aligned}$$

Since the support of $\mu_i(\cdot|p_t)$ is a subset of p_t^+ we have $\rho_t = p_t$ for all ρ in the support and

$$\begin{aligned}\dot{q}_i(t|\phi_i, p_t) &= \gamma \quad \text{if } p(t) < \bar{p} \\ \dot{q}_i(t|\phi_i, p_t) &= -\gamma \quad \text{if } p(t) > \bar{p}\end{aligned}$$

Hence the price path p_ϕ generated by ϕ is equal to the unique equilibrium price path p^* by Proposition 4. \square

Proof of Proposition 6. Let $\tau = \gamma^{-1} |f^{-1}(\bar{p}) - f^{-1}(p(0))|$. Let $\phi \in \Phi$ such that for every feasible history $p_t \in \mathcal{H}_f$ we have

$$\phi_i(p_t) = \begin{cases} \bar{p} - p(t) & \text{if } t < \tau \\ \gamma & \text{if } t \geq \tau \end{cases}$$

Then $p_\phi(\tau) = \bar{p}$ and $E_\tau \{p_\phi(T)\} > \bar{p}$. If $r < \beta$ then by Lemma 3 the discounted future economic rent under the price path p_ϕ at time τ satisfies

$$\begin{aligned} \lambda(\tau|p_\phi) &= \alpha\bar{p} + (1 - \alpha) E_\tau \{p_\phi(T)\} - p_\phi(\tau) \quad \text{where } \alpha = \frac{r}{\beta} \in (0, 1) \\ &= \alpha\bar{p} + (1 - \alpha) E_\tau \{p_\phi(T)\} - \bar{p} \\ &= (1 - \alpha) [E_\tau \{p_\phi(T)\} - \bar{p}] > 0 \end{aligned}$$

Since $\lambda(t|p_\phi)$ is continuous in t there exists $\tau' > \tau$ such that $\lambda(t|p_\phi) > 0$ for all $t \in [\tau, \tau']$. For all $p_t \in \mathcal{H}_f$ let $p_t^\triangleright \in p_t^+$ denote the price path such that

$$\begin{aligned} p_t^\triangleright(s) &= p_t(s) \quad \text{if } s \in [0, t] \\ p_t^\triangleright(s) &= \text{sgn}(\bar{p} - p_t^\triangleright(s)) \gamma f'(f^{-1}(p_t^\triangleright(s))) \quad \text{if } s \geq t \end{aligned}$$

Let $\mu_i \in M_i$ such that for all $p_t \in \mathcal{H}_f$

$$\begin{aligned} \mu_i(p_\phi|p_t) &= 1 \quad \text{if } p_\phi \in p_t^+ \\ \mu_i(p_t^\triangleright|p_t) &= 1 \quad \text{if } p_\phi \notin p_t^+ \end{aligned}$$

Let $\varphi \in \Phi$ such that for all $p_t \in \mathcal{H}_f$

$$\varphi_i(p_t) = \begin{cases} \text{sgn}(\lambda(t|p_\phi)) \gamma & \text{if } p_\phi \in p_t^+ \\ \text{sgn}(\lambda(t|p_t^\triangleright)) \gamma & \text{otherwise} \end{cases}$$

Then agent i 's subjective expected payoff under μ_i conditional on p_t is

$$\bar{\pi}_i(\varphi_i|\mu_i, p_t) = \begin{cases} \pi_i(\varphi_i|p_\phi) & \text{if } p_\phi \in p_t^+ \\ \pi_i(\varphi_i|p_t^\triangleright) & \text{otherwise} \end{cases}$$

Hence μ_i rationalizes φ_i by Proposition 2, so φ is a rational strategy and p_φ is a rational price path. But $p_\varphi(\tau') \neq \bar{p} = p^*(\tau')$ since $p_\varphi(\tau) = \bar{p}$ and $\text{sgn}(\dot{p}_\varphi(t)) = \text{sgn}(\lambda(\tau|p_\phi)) > 0$ for all $t \in [\tau, \tau']$. \square

