Multi-battle contests over complementary battlefields

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Abstract

This paper investigates multi-battle contests where a finite number agents allocate competitive resources to compete over a finite number of divisible prizes. The share of each prize awarded to each agent is determined by an arbitrarily decisive battlefield outcome function. Prizes serve as complementary factors with constant elasticity. Such multi-battle contests are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive battlefield outcome functions. In contrast, conventional multi-battle contests have no pure strategy equilibrium if battlefield outcome functions are sufficiently decisive. These results suggest that complementarity can play an important role in stabilizing the distribution of competitive resources over strategic conflicts.

1 Introduction

An agent's marginal value for one resource often depends on her share of other resources. Ride hailing firms compete to recruit drivers and market their platform to riders.[1] The marginal revenue from recruiting an additional driver depends in part on the firm's success marketing their platform to riders. Social media platforms compete for both users and advertisers. The marginal revenue from an additional user depends in part on the firm's success in obtaining advertisers.[2] Military factions compete for both air supremacy and ground supremacy. The marginal increase in a faction's control over a contested region from additional air supremacy may depend in part on the faction's level of ground supremacy.[3] Pharmaceutical firms compete to convince both doctors and patients of their product's effectiveness.[4] The marginal revenue from persuading an additional patient may depend in part on the firm's success in convincing doctors.

This paper considers multi-battle contests where a finite number of agents compete over a finite number of battlefields. Each agent allocates a stock of competitive resources between multiple battlefields. In each battlefield, agents compete over a distinct divisible prize. The share of each prize awarded to each agent is given by an arbitrarily decisive battlefield outcome function. Prizes serve as complementary factors with constant elasticity of substitution. Such contests are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive outcome functions. In contrast, conventional multi-battle conflicts and Blotto games have no pure strategy Nash equilibrium if outcome functions are sufficiently decisive.

The existence of a unique pure strategy Nash equilibrium under arbitrarily decisive battlefield outcome functions depends crucially on the smoothly non-linear relationship between payoffs and battlefield outcomes. This smooth non-linearity is possible because battlefield outcomes determine shares of divisible prizes rather than probabilities of obtaining indivisible prizes. The

[1] Farris et al. (2014) discusses competition for drivers between ride sharing firms.
[2] Fulgoni and Lipsman (2014) describes complementary users and advertisers.
[3] Pirnie et al. (2005) discusses complementarity between air and ground supremacy.
[4] See Hurwitz and Caves (1988) for more on rent seeking by pharmaceutical firms.

term “divisible prize” has been used to describe the outcome of a conflict over divisible resources by several authors including König et al. (2017). Divisible prizes are important for applications in economic and military settings[5] where “victory and defeat, although polar opposites, are not binary. There are thousands of points along the scale that delineate degrees of success” (Bartholomees, 2008).

The unique Nash equilibrium is shown to be Pareto efficient over the set of feasible outcomes. Hence any non-equilibrium strategy profile that gives one agent a greater payoff than she earns in equilibrium will give some other agent a lower payoff than she earns in equilibrium. This immediately rules out the possibility of collusive strategy profiles that give every agent a higher payoff then she earns in equilibrium. While the Nash equilibrium is Pareto efficient, most other feasible outcomes are Pareto dominated, so the contest is not strictly competitive. Equilibrium payoffs are shown to be minimax payoffs in the two-agent case, so any deviation from equilibrium by one agent can be exploited by the other to obtain an above-equilibrium payoff. The Pareto efficiency of the unique Nash equilibrium and the presence of minimax payoffs in the two agent case provide significant barriers to collusion despite the lack of strict competitiveness.

The remainder of this paper is organized as follows. Section 2 discusses the related literature. Section 3 formally describes the multi-battle contest. Section 4 establishes key properties of the best response correspondence. Section 5 establishes the existence of a unique pure strategy Nash equilibrium. Section 6 discusses barriers to collusion under the unique Nash equilibrium. Section 7 concludes and discusses important implications of the results. All proofs are provided in Appendix A.

[5] See Gray et al. (2002), Biddle (2004), and Martel (2011) for more on the measurement of victory on along continuous dimensions.

2 Related Literature

Previous literature on multi-battle contests often considers prizes that are perfect substitutes. Friedman (1958) considers multi-battle contests where two firms make advertising expenditures to compete over sales in several distinct marketing areas. Robson (2005) investigates two-player multi-item contests between resource constrained agents where prizes are perfect substitutes and outcome functions are probabilistic. Roberson (2006) examines two-player blotto games with deterministic winner-take-all outcome functions. A survey of the multi-battle contest literature is provided by Kovenock and Roberson (2010).

Several authors consider specific instances of complementarity in multi-unit auctions. Englmaier et al. (2009) identify asymmetric equilibria in two-bidder auctions over three items where a single item is has no value by itself and three items are worth no more than two items. Szentes and Rosenthal (2003) identify symmetric mixed strategy equilibria in auctions over three items where the marginal value increases for the second item and decreases for the third item. The complementarity in these models has a "chopstick" structure, reflecting the idea that a single chopstick is of little use and three chopsticks is little better than two.

Kolmar and Rommeswinkel (2013) examine contests between teams of agents who exert complementary effort and face linear costs. Analogously, Rai and Sarin (2009) consider contests where each agent makes multiple complementary investments to compete over a single prize. The complementarity in both of these models is between effort levels rather than between prizes. Intuitively, this type of complementarity is like the complementarity between the left oar and the right oar when rowing a boat. If you only paddle on only one side then you tend to go in circles. Forward progress is most effectively obtained by paddling on both sides.

Skaperdas and Syropoulos (1998) consider single-battle two-agent contests with an invisible prize that exhibit complementarity with effort levels, such that an agent's valuation for the prize depends on her own effort level. Malueg and Yates (2006) considers single-battle contests with homogeneous outcome functions where agents have a common value for the single indivisible prize. Ferrarese (2018) considers a multi-agent generalization of Malueg and Yates (2006) where an agent's valuation for the single divisible prize depends on the prize shares obtained by others. In contrast, the present paper considers contests with complementarity between multiple divisible prizes such that an agent's marginal value for each prize depends on her share of other prizes.

Kovenock et al. (2017) consider two-player multi-battle conflicts with deterministic winner-take-all outcome functions where prizes are perfect complements for the defender and perfect substitutes for the attacker. The complementarity in this multi-battle contest has a "weakest-link" structure such that the attacker need only win one battle to win the overall contest. Clark and Konrad (2007) consider the case of two firms that compete in multiple simultaneous patent races with probabilistic outcome functions. These outcome functions are restricted to have unit decisiveness. Each individual patent has a linearly additive value. Obtaining all of the patents yields an additional monopoly rent. If each patent is secured by some firm, but neither firm secures all of the patents, then they split the monopoly rent. This monopoly rent is the source of complementarity in their model.

Kovenock et al. (2015) consider two-player four-battle contests where each player has three possible minimal winning sets consisting of two battlefields each. If a player wins both of the battles in at least one of their winning sets then they win the overall contest. Snyder (1989) considers a contest between two political parties who make campaign expenditures to compete for legislative seats where each party aims to maximize their probability of obtaining a majority. Duffy and Matros (2017) examine two-player probabilistic blotto games where players seek to obtain a majority share of the overall prize value. Deck et al. (2017) investigate two-player multi-battle conflicts with linear effort costs where players seek to obtain a majority share of the overall prize value. The complementarity in these models has a majoritarian structure where agents are rewarded for obtaining a majority of the prizes.

The complementary in the present paper has a constant-elasticity structure, which is possible because each battlefield outcome function determines the share of a divisible prize rather than the probability of obtaining an indivisible prize. If outcome functions are sufficiently decisive, conventional blotto games and multi-battle conflicts often have no pure strategy Nash equilibrium. Baye et al. (1994) show that Tullock contests where agents expend costly effort at a constant marginal cost have no pure strategy Nash equilibrium if the outcome function is sufficiently decisive. Arbatskaya and Mialon (2010) show that two-player multi-activity contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium if the outcome function is sufficiently decisive. Roberson (2006) notes that conventional blotto games have no pure strategy Nash equilibrium unless one player is strong enough to guarantee complete victory in every battlefield. In contrast, the multi-battle contests considered by this paper are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive outcome functions.

3 Complementary Battlefields

Consider a multi-battle conflict where $n \geq 2$ agents simultaneously allocate limited resources over $m \geq 2$ complementary battlefields. Let $N = \{1, \ldots, n\}$ denote the set of agents and $B = \{1, \ldots, m\}$ denote the set of battlefields. Agent $i \in N$ is endowed with a stock $w_i \in \mathbb{R}_{++}$ of competitive resources. Let $x_{ib} \in \mathbb{R}_+$ denote the quantity of competitive resources that agent i allocates to battlefield b. The strategy $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}^m_+$ employed by agent i must satisfy the budget constraint $\sum_{k=1}^m x_{ib} = w_i$. Let $X_i = \left\{x_i \in \mathbb{R}^m_+ : \sum_{k=1}^m x_{ib} = w_i\right\}$ denote agent i's strategy set and let $X = \prod_{i \in N} X_i$ denote the set of strategy profiles.

In each battlefield, agents compete over a divisible prize. Agent i's share y_{ib} of prize b is given by the battlefield outcome function $\gamma_b : \mathbb{R}^n \to \triangle_{n-1}$ which is assumed to be continuous, homogeneous of degree zero, and independent from irrelevant alternatives such that if $x'_{kb} = 0$ and $x'_{jb} = x_{jb}$ for $j \in N \backslash \{i, k\}$ then

$$\gamma_{bi}(x') = \frac{\gamma_{bi}(x)}{1 - \gamma_{bk}(x)} \tag{1}$$

Agent i's share of prize b is assumed to be increasing in her allocation to battlefield b and decreasing in the allocations of other agents to battlefield b. The characterization of Clark and Riis (1998) then implies that if $x_{jb} > 0$ then agent i's share of prize b must be given by

$$y_{ib} = \gamma_{bi}(x) = \frac{\mu_i x_{ib}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a} \tag{2}$$

where $\mu \in \mathbb{R}_{++}^n$. If zero competitive resources are allocated to some battlefield b, then agent i's share y_{ib} of prize b is given by $\gamma_{bi}(x) = \mu_i / \sum_{j=1}^{n} \mu_j$. The parameter $a \in \mathbb{R}_{++}$ denotes the decisiveness of the battlefield outcome function. In the limit as $a \to \infty$ the entirety of prize b is awarded to the agent who allocates the most resources to battlefield b. Conversely, in the limit as $a \to 0$ prize b is equally divided over all the competitors in battlefield b. Agent i's battlefield outcome vector is given by $y_i = (y_{i1}, \ldots, y_{im}) \in \mathbb{R}_{+}^m$. Each of the m factors serves as a complementary input to agent i's payoff, which exhibits constant elasticity of substitution. If $y_i \notin \mathbb{R}_{++}^m$ then agent i's payoff is given by $\pi_i(y_i) = 0$.[6] Otherwise agent i's payoff π_i is given by[7]

$$\pi_i(y_i) = \left(\sum_{b=1}^{m} v_b y_{ib}^{-c_i} \right)^{-\frac{1}{c_i}} \tag{3}$$

The share parameter $v_b \in \mathbb{R}_{++}$ denotes the relative value of prize b. The sum of all m share parameters is given by $\sum_{b=1}^{m} v_b = 1$ without loss of generality since

$$\left(\sum_{b=1}^{m} \lambda v_b y_{ib}^{-c_i} \right)^{-\frac{1}{c_i}} = \lambda^{-\frac{1}{c}} \left(\sum_{b=1}^{m} v_b y_{ib}^{-c_i} \right)^{-\frac{1}{c_i}} = \lambda^{-\frac{1}{c_i}} \pi_i \tag{4}$$

The degree of complementary between battlefields for agent i is given by $c_i \in \mathbb{R}_{+}$. In the limit as $c_i \to \infty$, all m prizes are perfect complements and agent i's payoff is given by $\pi_i(y_i) = \min\{y_{i1}, \ldots, y_{im}\}$. Conversely, in the limit as $c_i \to 0$, the payoff to agent i takes the Cobb-Douglas form[8] $\pi_i(y_i) = \prod_{b=1}^{m} y_{ib}^{v_b}$. Here $c_i \in \mathbb{R}_{+}$ includes the case where $c_i = 0$ but does

[6]As shown on page 19, continuity requires that $\pi_i(y_i) = 0$ for all $y_i \notin \mathbb{R}_{++}^m$.
[7]See Uzawa (1962) for details regarding the necessity of this functional form.
[8]See Saito (2012) for a proof of convergence to Cobb-Douglas as $c_i \to 0$.

not include the limiting case of perfect complements where $c_i \to \infty$. As illustrated by Example 1, the Nash equilibrium need not be unique in the limiting case of perfect complements.

Example 1. Consider the case of two symmetrically endowed agents and two prizes in the limiting case of perfect complements where $c_i \to \infty$. If $x_1 = x_2 = (\theta, w_i - \theta)$ with $\theta \in (0, w_i)$ then $y_{ib} = \mu_i / (\mu_i + \mu_j)$. Both agents are best responding because any unilateral deviation would give the deviator less of at least one prize. If prizes are perfect complements, such a deviation would be unprofitable.

4 The Best Response

Agent i's payoff is continuous in her prize shares because the CES aggregator function is continuous. By (2) her share of prize b is continuous in her allocation if any agent allocates a non-zero quantity of resources to battlefield b. Consequently, agent i's payoff is continuous in her allocation over the interior of her strategy set. As illustrated by Example 2, if all n agents allocate zero resources to battlefield b then agent i can obtain the entirety of prize b by reallocating an arbitrarily small portion of her resources to it.

Example 2. Consider a simple contest with two players and two battlefields where $a = 1$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, $w = \mu = c = (1, 1)$. Suppose that both players allocate all of their resources to battlefield 1, so $x_1 = x_2 = (1, 0)$. Then agent 1's outcome profile is given by $y_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the payoff to agent 1 is given by $\pi_1 = \frac{1}{2}$. If agent 1 reallocates a small portion ε of her resources from battlefield 1 to battlefield 2 then her outcome vector will equal $y_1' = \left(\frac{1-\varepsilon}{2-\varepsilon}, 1\right)$ and her payoff will equal $\pi_1' = \left(\frac{1}{2}\left(\frac{2-\varepsilon}{1-\varepsilon}\right) + \frac{1}{2}\right)^{-1}$. Taking the limit as ε converges to zero obtains $\lim_{\varepsilon \to 0} \pi_1' = \frac{2}{3} > \frac{1}{2} = \pi_1$.

Proposition 1 states that agent i's payoff is strictly quasiconcave in her allocation x_i over the interior of her strategy set. Since her payoff is also continuous over this region, first order conditions are sufficient for the maximization of her payoff over the interior of her strategy set.

Proposition 1. *Agent i's payoff* π_i *is strictly quasiconcave over* $x_i \in \mathbb{R}^n_{++}$.

Proof. See appendix on page 19. □

Strict quasiconcavity holds for $c_i > 0$ because agent i's payoff is then a strictly increasing function of the strictly concave function

$$g_i(x_i) = -\frac{1}{c_i}\sum_{b=1}^{m} v_b \gamma_{bi}(x)^{-c_i} \tag{5}$$

Similarly, strict quasiconcavity holds for $c_i = 0$ because agent i's payoff is then a strictly increasing function of the strictly concave function

$$\log \pi_i = \sum_{b=1}^{m} v_b \log y_{ib} \tag{6}$$

The next proposition states that every resource allocation on the boundary of agent i's strategy set yields a strictly lower payoff than some other allocation in the interior of her strategy set. Thus agent i's best response always lies in the interior of her strategy set. Since agent i's payoff is strictly quasiconcave over the interior of her strategy set, she cannot have multiple best responses as a convex combination between any two distinct best responses would yield a larger payoff.

Proposition 2. *For every strategy profile* $x \in X$ *such that* $x_{ib} = 0$ *for some* b *there exits some alternative strategy* $x_i' \in X_i$ *such that* $\pi_i(x_i', x_{-i}) > \pi_i(x)$.

Proof. See appendix on page 20. □

Proposition 2 states that every best response lies in the interior of the strategy set. Agent i would obtain zero share of prize b if she allocates zero resources to the battlefield b but some other agent allocates a positive amount. Conversely, if none of the other agents allocate resources to battlefield b, then she could obtain the entirety of prize b by allocating an arbitrarily small amount of resources to it. Proposition 3 provides the first order conditions for the maximization of agent i's payoff under which agent i equalizes the marginal benefit of competitive resources in each battlefield.

Proposition 3. *A strategy $x_i \in X_i$ maximizes agent i's payoff π_i if and only if for all battlefields b and k we have*

$$\frac{v_b\left(1-y_{ib}\right)}{y_{ib}^{c_i}x_{ib}} = \frac{v_k\left(1-y_{ik}\right)}{y_{ik}^{c_i}x_{ik}} \tag{7}$$

Proof. See appendix on page 20. □

These first order conditions on agent i's allocation are both necessary and sufficient for payoff maximization because best responses are always unique and always lie in the interior of the strategy set. If agent i's marginal payoff from additional competitive resources in battlefield k was higher than her marginal payoff from additional competitive resources in battlefield b then agent i could achieve a higher payoff by reallocating resources from battlefield b to battlefield k. Rearranging the first order conditions to isolate the allocation ratio x_{ib}/x_{ik} yields

$$\frac{x_{ib}}{x_{ik}} = \frac{v_b y_{ib}^{-c_i}\left(1-y_{ib}\right)}{v_k y_{ik}^{-c_i}\left(1-y_{ik}\right)} \tag{8}$$

Since prizes are net complements, the parameter c_i is positive, so the right hand side of equation (8) is decreasing in y_{ib} and increasing in y_{ik}. If agent i is best responding and her share of prize b is larger than her share of prize k, then her allocation ratio x_{ib}/x_{ik} must be less than the corresponding share parameter ratio v_b/v_k. Conversely, if y_{ib} is smaller than y_{ik} then x_{ib}/x_{ik} must be greater than v_b/v_k. These results guarantee the uniqueness of the best response but they do note guarantee its existence. Example 3 illustrates how a best response can fail to exist.

Example 3. Consider a contest with two agents and two battlefields where $a = 1$, $c = 1$, $\mu_1 = \mu_2$, and $v_1 = v_2$. Suppose agent 1 allocates all of her resources to battlefield 1 such that $x_1 = (0, 1)$. Then agent 2 can obtain the entirety of prize 2 by allocating an arbitrarily small portion of her resources to battlefield 2. Hence $\pi_2(x_1, x_2) < \pi_2(x_1, x_2')$ where $x_2 = (1-\varepsilon, \varepsilon)$ and $x_2' = \left(1-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ for all $\varepsilon \in (0,1)$. Hence no interior strategy can be a best response for agent 2. Yet Proposition 2 states that any best response must lie in the interior of the strategy space, so agent 2 has no best response against $x_1 = (0,1)$.

5 The Nash Equilibrium

The marginal value of a small increase in agent i's share of prize b depends on her share of the other prizes. Complementarity between prizes incentivizes agent i to allocate relatively more competitive resources to battlefields where she is relatively less successful. As a result of this complementarity, Proposition 4 states that agent i must receive the same share of each prize in equilibrium.

Proposition 4. *In every pure strategy Nash equilibrium, $y_{ib} = y_{ik}$ for every agent i and all battlefields b and k.*

Proof. See appendix on page 21. □

If agent i's share of prize b is larger than her share of prize k then some other agent j must have a larger share of prize k than prize b, so agent i's allocation ratio x_{ib}/x_{ik} must be greater than agent j's allocation ratio x_{jb}/x_{jk} by equation (2). But if both agents are best responding then equation (8) implies that agent i's allocation ratio must be lower than agent j's allocation ratio. Hence agent i must obtain the same share of each prize in equilibrium, so equation (8) implies that her equilibrium allocation ratio must equal the corresponding share parameter ratio. Proposition 5 characterizes the resulting Nash equilibrium allocation profile.

Proposition 5. *The unique pure strategy Nash equilibrium is $x_{ib}^* = w_i v_b$.*

Proof. See appendix on page 21. □

Surprisingly, the Nash equilibrium strategy profile depends on neither the level of complementarity between prizes nor on the decisiveness of the battlefield outcome function. Yet the first order conditions imply that agent i's best response generally depends on both these parameters. Since $y_{ib} = y_{ik}$ in equilibrium, equation (8) reduces to

$$\frac{x_{ib}}{x_{ik}} = \frac{v_b}{v_k} \tag{9}$$

Since the terms involving a and c_i cancel out, neither the level of decisiveness nor the degree of complementarity effect the equilibrium allocation ratio. In contrast, conventional multi-battle contests and blotto games generally have no pure strategy Nash equilibrium if the battlefield outcome function is sufficiently decisive. Arbatskaya and Mialon (2010) show that multi-activity contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium under sufficiently decisive outcome functions. Roberson (2006) notes that conventional blotto games have no pure strategy Nash equilibrium unless one player is strong enough to guarantee victory in every battlefield.

If the battlefield outcome function (2) is highly decisive then agent i can obtain almost the entirety of prize b by allocating slightly more resources to battlefield b than her competitors. Yet as illustrated by Example 4, the presence of net complementarity between prizes makes such deviations from equilibrium unprofitable. In the absence of this complementarity, such deviations would be profitable under highly decisive battlefield outcome functions.

Example 4. Consider a contest with two players and two battlefields such that $w = \mu = c = (1, 1)$, and $v = \left(\frac{1}{4}, \frac{3}{4}\right)$. If each player allocates resources to each battlefield in proportion to the share parameters such that $x_1 = x_2 = v$ then each player obtains half of each prize and the outcome vectors are given by $y_1 = y_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ so payoffs are given by $\pi_1 = \pi_2 = \frac{1}{2}$. By equation (8), both agents are best responding. Now suppose agent 1's payoff is instead linear in her prize shares such that $\pi_i(x) = \frac{1}{4}y_{i1} + \frac{3}{4}y_{i2}$. In this case, if agent 1 employs the alternative strategy $x_1' = (0, 1)$ then for highly decisive battlefields her limiting payoff is given by $\lim_{a \to \infty} \pi_1(x_1', x_2) = \frac{3}{4} > \frac{1}{2} =$

$\lim_{a \to \infty} \pi_1 (x_1, x_2)$.

6 Barriers to Collusion

Proposition 6 states that agent i's Nash equilibrium payoff is a function of her endowment and the decisiveness of the battlefield outcome function. Although the equilibrium strategy profile is unresponsive to the decisiveness parameter, the equilibrium payoffs are shown to exhibit greater sensitivity to initial endowments under higher decisiveness levels. As the battlefield outcome function becomes increasingly decisive, the agent with the largest initial endowment earns an increasingly large share of the equilibrium payoffs. Conversely, decreasing the decisiveness parameter decreases the share of equilibrium payoffs earned by the agent with the largest initial endowment without distorting equilibrium allocation behavior.

Proposition 6. *The unique Nash equilibrium payoff to agent i is given by*

$$\pi_i^* = \frac{\mu_i w_i^a}{\sum_{\ell=1}^n \mu_\ell w_\ell^a} \tag{10}$$

Proof. See appendix on page 22. □

Agent i's unique Nash equilibrium payoffs are proportional to $\mu_i w_i^a$ because equilibrium allocations to each battlefield are given by $x_i = w_i v_b$ and her share of prize b is proportional to $\mu_i x_{ib}^a$. Proposition 7 states that the Nash equilibrium maximizes the total payoff to all n agents, so the equilibrium strategy profile is Pareto efficient over the set of feasible outcomes. Any non-equilibrium strategy profile that gives one agent a greater payoff than she earns in equilibrium must give some other agent a lower payoff than she earns in equilibrium. This immediately rules out the possibility of collusive strategies that give each agent a higher payoff then she earns in equilibrium.

Proposition 7. *The maximum total payoff to all n agents over all feasible* *strategy profiles $x \in X$ is given by*

$$\max_{x \in X} \sum_{i=1}^{n} \pi_i(x) = 1$$

Proof. See appendix on page 22. □

The unique Nash equilibrium maximizes the total payoff to all n agents because the prizes are net complements and the equilibrium equalizes agent i's share of each prize. The resulting Pareto efficiency of the Nash equilibrium immediately rules out the possibility of collusive strategies that give each agent a higher payoff then she earns in equilibrium. Proposition 8 states that an agent can always obtain an above-equilibrium payoff in the two agent case if her opponent employs a non-equilibrium strategy. Hence the Nash equilibrium payoffs are also the minimax payoffs in the two agent case because agent i can always obtain at least her Nash equilibrium payoff by mirroring agent j's strategy.

Proposition 8. *If agent j employs a non-equilibrium strategy and $n = 2$ then agent i can obtain an above-equilibrium payoff.*

Proof. See appendix on page 23. □

As illustrated by Example 5, the aggregate total payoff depends on the allocation profile. Although the equilibrium outcome is Pareto efficient, other feasible outcomes are Pareto dominated. Even in the two agent case, both the "size of the pie" and the "division of the pie" depend on the allocation strategies, so the contest is not strictly competitive. Together, the Pareto efficiency of the unique Nash equilibrium and the presence of minimax equilibrium payoffs in the two agent case provide strong barriers to collusion despite the lack of strict competitiveness.

Example 5. Consider a contest with two players and two battlefields where $a = 1$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $w = \mu = c = (1, 1)$. If both agents employ their equilibrium strategies we have $x_{ib} = \frac{1}{2}$ and $y_{ib} = \frac{1}{2}$ for every agent i and every battlefield b. Hence the payoff to each agent is given by $\pi_i = \frac{1}{2}$, so the aggregate payoff is given by $\pi_1 + \pi_2 = 1$. Now suppose instead that agent 1 employs the non-equilibrium strategy $x'_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and agent 2 employs the non-equilibrium strategy $x'_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$. In this case agent 1's outcome profile is $y'_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and agent 2's outcome profile is $y'_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$. Hence the payoff to each agent is given by $\pi'_i = \frac{4}{9}$, and the total payoff to both agents is $\pi'_1 + \pi'_2 = \frac{8}{9} < 1 = \pi_1 + \pi_2$.

7 Conclusion

This paper considers multi-battle conflicts where n agents allocate competitive resources to compete over m complementary battlefields. The share of each prize awarded to each agent is given by an arbitrarily decisive battlefield outcome function. Payoffs exhibit nonlinear dependence on battlefield outcomes with an arbitrary degree of complementarity. These contests are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive outcome functions. In contrast, conventional multi-battle contests have no pure strategy Nash equilibrium if outcome functions are sufficiently decisive. These results suggest that complementarity can play an important role in stabilizing the distribution of competitive resources.

The unique Nash equilibrium is shown to be Pareto efficient over the set of feasible outcomes. This immediately rules out the possibility of collusive strategy profiles that give every agent a higher payoff then she earns in equilibrium. In the two agent case, equilibrium payoffs are shown to be minimax payoffs, so any deviation from equilibrium by one agent can be exploited by the other to obtain an above-equilibrium payoff. Although the Nash equilibrium is Pareto efficient, most other feasible outcomes are Pareto dominated, so the contest is not strictly competitive. Together, Pareto efficiency and the presence of minimax payoffs in the two agent case provide significant barriers to collusion despite the lack of strict competitiveness.

Further research is necessary to understand several important generalizations of the present model. If prizes were net substitutes instead of net complements then payoffs might fail to be quasiconcave and the first order conditions provided by Proposition 3 might fail to characterize the best response. Future research should characterize Nash equilibria for the case of net substitutes. The present battlefields exhibit distinct valuation levels but share a symmetric decisiveness level. Future research should consider the case of asymmetric decisiveness across battlefields. Agents presently have symmetric valuations for battlefield b. Future research should consider the case where each agent has a distinct valuation for battlefield b.

References

Maria Arbatskaya and Hugo M Mialon. Multi-activity contests. *Economic Theory*, 43(1):23–43, 2010.

J. Boone Bartholomees. Theory of victory. *Parameters*, pages 25–36, 2008.

Michael R Baye, Dan Kovenock, and Casper G De Vries. The solution to the tullock rent-seeking game when r> 2: Mixed-strategy equilibria and mean dissipation rates. *Public Choice*, 81(3-4):363–380, 1994.

Stephen D Biddle. *Military power: Explaining victory and defeat in modern battle*. Princeton University Press, 2004.

Derek J. Clark and Kai A. Konrad. Asymmetric conflict: Weakest link against best shot. *Journal of Conflict Resolution*, 51(3):457–469, 2007.

Derek J Clark and Christian Riis. Contest success functions: an extension. *Economic Theory*, 11(1):201–204, 1998.

Cary Deck, Sudipta Sarangi, and Matt Wiser. Electoral college: A multibattle contest with complementarities. 2017. Working Paper.

John Duffy and Alexander Matros. Stochastic asymmetric Blotto games: An experimental study. *Journal of Economic Behavior & Organization*, 139 (C):88–105, 2017.

Florian Englmaier, Pablo Guillen, Loreto Llorente, Sander Onderstal, and Rupert Sausgruber. The chopstick auction: a study of the exposure problem in multi-unit auctions. *International Journal of Industrial Organization*, 27(2):286–291, 2009.

Paul Farris, Gerry Yemen, Virginia Weiler, and Kusum L Ailawadi. Uber pricing strategies and marketing communications. Darden Case No. UVA-M-0871, 2014.

Walter Ferrarese. Equilibrium effort in games with homogeneous production functions and homogeneous valuation. 2018. Working Paper.

Lawrence Friedman. Game-theory models in the allocation of advertising expenditures. *Operations research*, 6(5):699–709, 1958.

Gian Fulgoni and Andrew Lipsman. Digital game changers: how social media will help usher in the era of mobile and multi-platform campaign-effectiveness measurement. *Journal of Advertising Research*, 54(1):11–16, 2014.

Colin Gray et al. *Defining and achieving decisive victory*. DIANE Publishing, 2002.

Mark A Hurwitz and Richard E Caves. Persuasion or information? promotion and the shares of brand name and generic pharmaceuticals. *The journal of law and economics*, 31(2):299–320, 1988.

Martin Kolmar and Hendrik Rommeswinkel. Contests with group-specific public goods and complementarities in efforts. *Journal of Economic Behavior & Organization*, 89:9–22, 2013.

Michael D König, Dominic Rohner, Mathias Thoenig, and Fabrizio Zilibotti. Networks in conflict: Theory and evidence from the great war of africa. *Econometrica*, 85(4):1093–1132, 2017.

Dan Kovenock and Brian Roberson. *Conflicts with multiple battlefields*, chapter Conflicts with multiple battlefields. Oxford University Press, 2010.

Dan Kovenock, Sudipta Sarangi, and Matt Wiser. All-pay 2x2 hex: a multi-battle contest with complementarities. *International Journal of Game Theory*, 44(3):571–597, Aug 2015.

Dan Kovenock, Brian Roberson, and Roman M. Sheremeta. The attack and defense of weakest-link networks. 2017. ESI Working Papers 17-19.

David A Malueg and Andrew J Yates. Equilibria in rent-seeking contests with homogeneous success functions. *Economic Theory*, 27(3):719–727, 2006.

William C Martel. *Victory in war: foundations of modern strategy*. Cambridge University Press, 2011.

Bruce R Pirnie, Alan Vick, Adam Grissom, Karl P Mueller, and David T Orletsky. Beyond close air support. forging a new air-ground partnership. Technical report, Rand Corporation, Santa Monica, CA, 2005.

Birendra K Rai and Rajiv Sarin. Generalized contest success functions. *Economic Theory*, 40(1):139–149, 2009.

Brian Roberson. The colonel blotto game. *Economic Theory*, 29(1):1–24, 2006.

Alex Robson. Multi-Item Contests. ANU Working Papers in Economics and Econometrics 2005-446, Australian National University, College of Business and Economics, School of Economics, 2005.

Tetsuya Saito. How do we get cobb-douglas and leontief functions from ces functions: A lecture note on discrete and continuum differentiated object models. *Journal of Industrial Organization Education*, 6(1), 2012.

Stergios Skaperdas and Constantinos Syropoulos. Complementarity in contests. *European Journal of Political Economy*, 14(4):667–684, 1998.

James M Snyder. Election goals and the allocation of campaign resources. *Econometrica: Journal of the Econometric Society*, pages 637–660, 1989.

Balazs Szentes and Robert W. Rosenthal. Beyond chopsticks: Symmetric equilibria in majority auction games. *Games and Economic Behavior*, 45 (2):278 – 295, 2003. Special Issue in Honor of Robert W. Rosenthal.

Hirofumi Uzawa. Production functions with constant elasticities of substitution. *The Review of Economic Studies*, 29(4):291–299, 1962.

A Proofs

Proof of Payoff Continuity. If $y_i \in \mathbb{R}^m_{++}$then the payoff to agent i is given by

$$\pi_i(y_i) = \left(\sum_{b=1}^{m} v_b y_{ib}^{-c_i}\right)^{-\frac{1}{c_i}} = \frac{1}{\left(\sum_{b=1}^{m} \frac{v_b}{y_{ib}^{c_i}}\right)^{c_i}} \tag{11}$$

Since $c_i > 0$ the fraction $v_b / y_{ib}^{c_i}$ increases without bound as y_{ib} converges to zero, so the sum $\left(\sum_{b=1}^{m} \frac{v_b}{y_{ib}^{c_i}}\right)^{c_i}$ increases without bound as as y_{ib} converges to zero. Hence $\pi_{ib}(y_i)$ converges to zero as y_{ib} converges to zero, so agent i's payoff π_i is continuous in her success vector y_i $\square$

Proof of Proposition 1. Suppose $c_i > 0$. Let g_i denote an increasing function of π_i given by

$$g_i = -\frac{1}{c\pi_i^{c_i}} = -\frac{1}{c_i}\sum_{b=1}^{m} v_b y_{ib}^{-c_i} \tag{12}$$

Differentiating y_{ib} with respect to x_{ib} yields

$$\begin{aligned} \frac{\partial y_{ib}}{\partial x_{ib}} &= \frac{\partial}{\partial x_{ib}}\left[\frac{\mu_i x_{ib}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right] = \frac{\sum_{j\neq i} \mu_j x_{jb}^a}{\left(\sum_{j=1}^{n} \mu_j x_{jb}^a\right)^2} a\mu_i x_{ib}^{a-1} \\ &= \frac{a}{x_{ib}}\left(\frac{\sum_{j\neq i} \mu_j x_{jb}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right)\left(\frac{\mu_i x_{ib}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right) = \frac{a(1-y_{ib})\, y_{ib}}{x_{ib}} \end{aligned}$$

So differentiating g_i with respect to x_i yields

$$\frac{\partial g_i}{\partial x_{ib}} = \frac{\partial g_i}{\partial y_{ib}}\frac{\partial y_{ib}}{\partial x_{ib}} = \frac{v_b}{y_{ib}^{1+c_i}}\frac{a(1-y_{ib})\, y_{ib}}{x_{ib}} = \frac{a v_b (1-y_{ib})}{y_{ib}^{c_i} x_{ib}} \tag{13}$$

Since the numerator of (13) is decreasing in x_{ib} and the denominator is increasing in x_{ib} we have $\frac{\partial^2 g_i}{\partial x_{ib}^2} < 0$. Since (13) is constant in x_{ih} for all $h \neq b$, the mixed second order partial derivatives are given by $\frac{\partial^2 g_i}{\partial x_{ib}\partial x_{ih}} = 0$. Thus the matrix of second order partial derivatives is negative definite, so g_i is strictly concave in x_i. Hence π_i is strictly quasiconcave in x_i for $c_i > 0$ since g_i is a strictly increasing function of π_i. If $c_i = 0$ then the payoff to agent i

is given by

$$\pi_i = \prod_{b=1}^{m} y_{ib}^{v_b} \tag{14}$$

Taking the logarithm of both sides obtains

$$\log \pi_i = \sum_{b=1}^{m} v_b \log y_{ib} \tag{15}$$

Differentiating with respect to x_{ib} yields

$$\frac{\partial}{\partial x_{ib}} \Big[\log \pi_i \Big] \quad = \quad \frac{v_b}{y_{ib}} \frac{\partial y_{ib}}{\partial x_{ib}} = \frac{a\left(1 - y_{ib}\right)}{x_{ib}} \tag{16}$$

Thus $\frac{\partial^2 \log \pi_i}{\partial x_{ib}^2} < 0$ and $\frac{\partial^2 \log \pi_i}{\partial x_{ib} \partial x_{ih}} = 0$ for $b \neq h$. Hence π_i is strictly quasiconcave in x_i for $c_i = 0$. □

Proof of Proposition 2. Let $x \in X$ such that $x_{ib} = 0$. Now consider the alternative strategy $\hat{x}_i \in X_i$ such that

$$\hat{x}_{ik} = \varepsilon \frac{w_i}{m} + (1 - \varepsilon) x_{ik} \tag{17}$$

If $x_{jb} > 0$ for some $j \neq i$ then $\pi_i(x) = 0 < \pi_i(\hat{x}_i, x_{-i})$. Alternatively, if $x_{jb} = 0$ for all j then $\gamma_{bi}(x) = \mu_i / \sum_{j=1}^{n} \mu_j < 1 = y_{ib}(\hat{x}_i, x_{-i})$ for all $\varepsilon > 0$. Since $x_{jb} = 0$ for all $j \in N$ there exists at least one battlefield $h \in B$ such that $x_{jh} > 0$ for some $j \neq i$. Then the limiting value of $\gamma_{hi}(\hat{x}_i, x_{-i})$ as ε approaches zero from above is given by

$$\lim_{\varepsilon \downarrow 0} \gamma_{hi}(\hat{x}_i, x_{-i}) = \lim_{\varepsilon \downarrow 0} \frac{\mu_i x_{ih}}{\sum_{j=1}^{n} \mu_j x_{jh}^a} = y_{ih}(x) \tag{18}$$

Hence $\pi_i(\hat{x}_i, x_{-i}) > \pi_i(x)$ for some $\varepsilon > 0$ since π_i is continuous over $y_i \in \mathbb{R}_{++}^m$. □

Proof of Proposition 3. Suppose that x_i is a best response for agent i. By Proposition 2, x_i must lie in the interior of agent i's strategy set. Hence agent i's marginal benefit from increasing her allocation to battlefield i must equal her marginal benefit from increasing her allocation to battlefield j. Since agent i's payoff is continuous over the interior of her strategy set we have

$$\frac{\partial \pi_i}{\partial x_{ib}} = \frac{\partial \pi_i}{\partial x_{ik}} \tag{19}$$

$$\frac{v_b (1 - y_{ib})}{y_{ib}^{c_i} x_{ib}} = \frac{v_k (1 - y_{ik})}{y_{ik}^{c_i} x_{ik}} \tag{20}$$

Conversely, if x_i satisfies these first order conditions then it must lie in the interior of agent i's strategy set. By Proposition 2 agent i's best response must lie in the interior of her strategy set. By Proposition 1 agent i's payoff π_i is strictly quasiconcave over this region. Since her payoff is also continuous in her strategy over this region, the first order conditions on x_i are sufficient for maximization of agent i's payoff over her strategy set. □

Proof of Proposition 4. If $y_{ib} < y_{ik}$ then we have

$$\sum_{\ell \neq i} y_{\ell b} = (1 - y_{ib}) > (1 - y_{ik}) = \sum_{\ell \neq i} y_{\ell k} \tag{21}$$

Hence there exists $j \neq i$ such that $y_{jk} < y_{jb}$ and

$$y_{ib} y_{jk} < y_{ik} y_{jb} \tag{22}$$

$$\left(\frac{\mu_i x_{ib}^a}{Z_b}\right)\left(\frac{\mu_j x_{jk}^a}{Z_k}\right) < \left(\frac{\mu_i x_{ik}^a}{Z_k}\right)\left(\frac{\mu_j x_{jb}^a}{Z_b}\right) \quad \text{where } Z_b = \sum_{\ell=1}^{n} \mu_\ell x_{\ell b}^a \tag{23}$$

$$x_{ib} x_{jk} < x_{ik} x_{jb} \tag{24}$$

By Proposition 3 the first order conditions on x_i can be written as

$$\frac{x_{ib}}{x_{ik}} = \frac{v_b (1 - y_{ib}) \, y_{ib}^{-c_i}}{v_k (1 - y_{ik}) \, y_{ik}^{-c_i}} \tag{25}$$

If x is a Nash equilibrium then by equation (25) we have

$$\begin{matrix} y_{ib} < y_{ik} \\ y_{jk} < y_{jb} \end{matrix} \implies \frac{x_{ib}}{x_{ik}} > \frac{v_b}{v_k} > \frac{x_{jb}}{x_{jk}} \implies x_{ib} x_{jk} > x_{ik} x_{jb} \tag{26}$$

But this contradicts equation (24). □

Proof of Proposition 5. By Proposition 4 if x is a Nash equilibrium strategy profile then for every agent i there exists $\bar{y}_i \in [0, 1]$ such that for every battlefield b agent i's share of prize b is given by $y_{ib} = \bar{y}_i$. Hence by Proposition 3 the necessary and sufficient first order conditions on x_i for the maximization of π_i are given by

$$\frac{v_b(1-\bar{y}_i)}{x_{ib}\bar{y}_i^{c_i}} = \frac{v_k(1-\bar{y}_i)}{x_{ik}\bar{y}_i^{c_i}} \tag{27}$$

$$\frac{v_b}{v_k} = \frac{x_{ib}}{x_{ik}} \tag{28}$$

Hence $x_{ib}v_k = x_{ik}v_b$ and summing over k obtains $x_{ib} = w_i v_b$. □

Proof of Proposition 6. By Proposition 4 if x is a Nash equilibrium strategy profile then for every agent i there exists $\bar{y}_i \in [0,1]$ such that for every battlefield b agent i's share of prize b is given by $y_{ib} = \bar{y}_i$. Hence the payoff to agent i can be written as

$$\pi_i = \left(\sum_{b=1}^{m} v_b \bar{y}_i^{-c_i}\right)^{-\frac{1}{c_i}} = \bar{y}_i \left(\sum_{b=1}^{m} v_b\right)^{-\frac{1}{c_i}}$$

Now since $\sum_{b=1}^{m} v_b = 1$ we have

$$\pi_i = \bar{y}_i = \frac{\mu_i w_i^a}{\sum_{\ell=1}^{n} \mu_\ell w_\ell^a} \tag{29}$$

□

Proof of Proposition 7. Let Y denote the set of all $y \in \mathbb{R}_+^{n\times m}$ such that for all battlefields $b \in B$ the sum of all prize shares is given by $\sum_{i=1}^{n} y_{ib} = 1$. Hence Y includes all feasible outcomes. Let g_i denote a strictly increasing function of π_i given by

$$g_i = -\frac{1}{c_i \pi_i^{c_i}} = -\frac{1}{c_i}\sum_{b=1}^{m} v_b y_{ib}^{-c_i} \tag{30}$$

For $\theta \in \triangle^{n-1}$ let G_θ denote a weighted sum of all g_i given by

$$G_\theta = \sum_{i=1}^{n} \theta_i g_i = -\sum_{i=1}^{n} \frac{1}{c_i} \sum_{b=1}^{m} \theta_i v_b y_{ib}^{-c_i} \tag{31}$$

Hence G_θ is increasing in π_i for each agent i. Now differentiating G_θ with respect to y_{ib} yields $\frac{\partial G_\theta}{\partial y_{ib}} = \frac{\theta_i v_b}{y_{ib}^{c_i+1}} > 0$ and twice differentiating G_θ with respect to y_{ib} yields $\frac{\partial^2 G_\theta}{\partial y_{ib}^2} = -(c_i+1)\frac{\theta_i v_b}{y_{ib}^{c_i+2}} < 0$. The cross partial derivatives are given by $\frac{\partial G}{\partial y_{ib}\partial y_{jb}} = 0$. Hence G_θ is strictly concave over $y_{Nb} = (y_{1b}, \ldots, y_{nb}) \in$

$\mathbb{R}^m_{++}$ so the first order conditions on y_{Nb} for the maximization of G_θ are given by

$$\frac{\theta_i v_b}{y_{ib}^{c_i+1}} = \frac{\partial G}{\partial y_{ib}} = \frac{\partial G}{\partial y_{jb}} = \frac{\theta_j v_b}{y_{jb}^{c_j+1}} \tag{32}$$

$$\frac{\theta_i}{\theta_j} = \frac{y_{ib}^{c_i+1}}{y_{jb}^{c_j+1}} \implies y_{ib} = \bar{y}_i \tag{33}$$

Thus if y maximizes G_θ over Y then the payoff to agent i satisfies $\pi_i = \bar{y}_i$. Now if $y \in Y$ maximizes the total payoff $\sum_{i=1}^n \pi_{ib}$ over Y then it is Pareto efficient over Y, so there exists some $\theta \in \triangle^{n-1}$ such that y maximizes G_θ over Y and the total payoff is given by

$$\sum_{i=1}^n \pi_i = \sum_{i=1}^n \bar{y}_i = 1$$

□

Proof of Proposition 8. Let x_j denote an allocation employed by agent j and suppose that agent i employs the allocation

$$x_{ib} = \frac{w_i x_{jb}}{w_j} \tag{34}$$

Then the share of prize b awarded to agent i is given by

$$y_{ib} = \frac{\mu_i x_{ib}^a}{\mu_i x_{ib}^a + \mu_j x_{jb}^a} = \frac{\mu_i w_i^a}{\mu_i w_i^a + \mu_j w_j^a} = \bar{y}_i \tag{35}$$

Hence the payoff to agent i is given by

$$\pi_i = \left(\sum_{b=1}^m v_b \bar{y}_i^{-c_i} \right)^{-\frac{1}{c_i}} = \bar{y}_i \left(\sum_{b=1}^m v_b \right)^{-\frac{1}{c_i}} = \frac{\mu_i w_i^a}{\mu_i w_i^a + \mu_j w_j^a} \tag{36}$$

Thus by Proposition 6 agent i can always obtain at least her unique Nash equilibrium payoff. Now if $x_{jb} \neq w_j v_b$ then the strategy given by Equation (34) does not satisfy the first order conditions for the maximization of agent i's payoff. Hence it is not a best response by Proposition 3, so there exists some alternative strategy that does better. □