The Reliability of Equilibrium in Conflicts over Complementary Factors

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Abstract

This paper experimentally investigates conflicts where competitors allocate resources between multiple contests to compete for shares of complementary factors. Each factor is divided among the competitors in proportion to a power function of their investment in the corresponding contest. In equilibrium, more responsive success functions produce stronger incentives to employ equilibrium strategies in response to equilibrium strategies. Out of equilibrium, best responses to nonequilibrium strategies are further from equilibrium predictions under more responsive success functions. Accordingly, the experimental design varies the responsiveness of the success function across treatment conditions. Consistent with nonequilibrium incentives, observed resource allocations were closer to equilibrium predictions under less responsive success functions. These results suggest that nonequilibrium incentives can influence the reliability of equilibrium predictions in conflicts over complementary factors.

Keywords: Conflict, Resource Allocation, Complementary Factors

JEL Classification: C72, C92, D74

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1 Introduction

Strategic conflicts frequently involve the allocation of resources between multiple domains of competition. Military conflicts often involve competition for control over both airspace and the ground below it. Ride sharing platforms compete for both riders and drivers. Different domains of competition often provide complementary inputs to a decision maker's overall objective. The value of increasing control over airspace may depend on a military faction's level of control over the ground below it (Pirnie et al., 2005). Similarly, the value of an additional rider to a ride sharing platform may depend on the platform's success in attracting drivers (Rochet and Tirole, 2003).

This paper reports an experimental study of conflicts where competitors allocate resources between multiple contests to compete for shares of divisible factors. Each factor is divided among the competitors in proportion to a power function of the resources they allocate to the corresponding contest. Each competitor aims to maximize an objective function that exhibits unitary elasticity between factor shares. In the unique Nash equilibrium, resource allocations are proportional to factor values.

Competitors who expect others to employ equilibrium strategies have stronger incentives to follow equilibrium predictions when success functions are more responsive to investment levels. Accordingly, one might expect observed resource allocations to approximate equilibrium predictions more closely under more responsive success functions. Yet best responses to nonequilibrium allocations deviate further from equilibrium under more responsive success functions, so one might expect observed resource allocations to approximate equilibrium predictions more closely under less responsive success functions.

To test these hypotheses, the experimental design varies the responsiveness of contest success functions across treatment conditions. The equilibrium predictions are identical in all treatment conditions. Consistent with best responses to nonequilibrium behavior, observed resource allocations were significantly closer to equilibrium under less responsive success functions. These results suggest that nonequilibrium incentives had a significant impact on the reliability of equilibrium predictions.

This paper contributes to the experimental literature on Blotto contests where competitors allocate limited resources to compete over multiple prizes. Much of the previous literature focuses on Blotto contests with indivisible prizes where resource allocations influence the probability of winning a given prize. Duffy and Matros (2017) find support for equilibrium predictions in stochastic Blotto contests with winner-take-all battles and majoritarian objective functions. Chowdhury et al. (2021) find that subjects over-allocate resources to battlefields with distinctive labels in stochastic Blotto contests with winner-take-all battles. In contrast, the present paper investigates of Blotto contests over shares of complementary factors.

There is also a significant body of experimental research on contests where agents compete for a single prize. Baik et al. (2020) experimentally identify a nonmonotonic relationship between budget constraints and average bids. Llorente-Saguer et al. (2023) report experimental support for theoretical predictions that bid-caps and tie-breaking rules can increase total expenditure in contests with heterogeneous contestants. A survey of this literature is provided by Cason et al. (2020).

The present paper contributes to the experimental literature investigating how nonequilibrium incentives influence the reliability of equilibrium predictions. Cason et al. (2014) observe behavior that is closer to equilibrium in evolutionarily stable rock-paper-scissors games. The conflicts investigated by the present study are evolutionarily stable in all treatments, so evolutionary stability does not explain the observed treatment effects. Chen and Gazzale (2004) observe behavior that is closer to equilibrium in supermodular compensation mechanisms. The conflicts investigated by the present study are supermodular in all treatments, so supermodularity does not explain the observed treatment effects. Stephenson (2022) observes behavior that is closer to equilibrium in school choice mechanisms with high frequency feedback. The present study provides feedback at the same frequency in all treatments, so feedback frequency does not explain the treatment effects observed in the present study.

2 Theory

Consider a conflict where two competitors simultaneously allocate a fixed budget between two contests. Let $x_{ik} \in \mathbb{R}_+$ denote the share of competitor i's resources allocated to contest k. As in the Blotto contest of Borel (1921), total resource investments are sunk before competitors allocate them between contests. Let X_i denote the set of allocations $x_i \in \mathbb{R}^2_+$ such that $x_{i1} + x_{i2} = 1$. The success function $y_{ik}(x)$ describes competitor i's share of factor k as a function of the allocation profile $x \in \mathbb{R}^{2\times 2}_+$. If $x_{1k} = x_{2k} = 0$, then factor k is divided evenly between the competitors, so $y_{ik}(x) = \frac{1}{2}$. If $x_{1k} + x_{2k} > 0$ then the success function takes the generalized Tullock (1980) form under which competitor i's share of factor k is proportional to a power function of their investment in contest k.

$$y_{ik}(x_i, x_j) = \frac{x_{ik}^{\alpha}}{x_{ik}^{\alpha} + x_{jk}^{\alpha}} \tag{1}$$

The parameter α describes the responsiveness of the success function $y_{ik}(x)$ to the investment levels x_{ik} and x_{jk}. If α is very large then nearly the entirety of factor k is awarded to the competitor who allocates the most resources to contest k. If α is very small then factor shares are largely insensitive to resource allocations. Let $v_k \in (0,1)$ denote the relative value of factor k such that $v_1 + v_2 = 1$. Factor shares are complementary inputs to competitor i's objective function $\pi_i : X \to \mathbb{R}$. If $y_{ik}(x) = 0$ then $\pi_i(x) = 0$. If $y_i(x) \in \mathbb{R}^2_{++}$ then competitor i's payoff exhibits unitary elasticity of substitution between factors.

$$\pi_i(x_i, x_j) = \beta \left(v_1 y_{i1}(x)^{-1} + v_2 y_{i2}(x)^{-1}\right)^{-1} \tag{2}$$

This is not a zero sum game because the total payoff $\pi_1(x) + \pi_2(x)$ varies with the strategy profile x. Consider the simple case where $v_1 = v_2 = \frac{1}{2}$ and $\alpha = 1$. If both competitors select identical resource allocations then $x_1 = x_2$ and $\pi_1(x) = \pi_1(x) = \frac{1}{2}\beta$. In this case, the total payoff is $\pi_1(x) + \pi_2(x) = \beta$. In contrast if $x_1 = \left(\frac{1}{4}, \frac{3}{4}\right)$ and $x_2 = \left(\frac{3}{4}, \frac{1}{4}\right)$ then $\pi_1(x) = \pi_2(x) = \frac{3}{8}\beta$. In this case, the total payoff is $\pi_1(x) + \pi_2(x) = \frac{3}{4}\beta$.

Strategic conflicts frequently involves competition over divisible factors. For example, military conflicts may involve competition for control over both airspace and the ground below it. The value of additional control over the airspace in a given region may depend on a military faction's level of control over the ground below it (Pirnie et al., 2005). Similarly, ride sharing platforms simultaneously compete for both riders and drivers. The value of an additional rider may depend on a firm's success in attracting drivers (Rochet and Tirole, 2003).

Competitor i's allocation is said to be a best response to competitor j's allocation if it maximizes competitor i's payoff, taking competitor j's allocation as given. An allocation profile x is said to be an equilibrium if each competitor best responds to the allocation selected by their opponent. Theorem 1 (Stephenson, 2024) says that there is a unique Nash equilibrium under which each competitor allocates resources to contest k in proportion to the value of prize k. Theorem 1 motivates Hypothesis 1 presented in Section 4.

Theorem 1. *There exists a unique Nash equilibrium where* $x_{ik} = v_k$.

Theorem 2 says that best responses to nonequilibrium strategies are farther from equilibrium under more responsive success functions. The distance between the equilibrium strategy and agent i's best response to a nonequilibrium strategy is increasing in the sensitivity level α. This theorem motivates Hypotheses 2 as presented in Section 4.

Theorem 2. *If* $0 \neq x_{jk} \neq v_k$ *then agent* i*'s best response* $x_i^*(x_j)$ *is unique and*

$$\frac{\partial \left| x_{ik}^*(x_j) - v_k \right|}{\partial \alpha} > 0 \tag{3}$$

Theorem 3 describes the relationship between the responsiveness of the success function and the strength of equilibrium incentives. It says that the incentive to employ an equilibrium strategy in response to an equilibrium strategy is stronger under more responsive success functions. This theorem motivates Hypotheses 3 as presented in Section 4.

Theorem 3. *If* $0 \neq x_{ik} \neq v_k$ *then*

$$\frac{\partial \left| \pi_i(v, v) - \pi_i(x_i, v) \right|}{\partial \alpha} > 0 \tag{4}$$

Figure 1 illustrates the best response correspondence for $\alpha = 1$ and $\alpha = 8$. The horizontal axis indicates competitor j's allocation to contest 1 and the vertical axis indicates competitor i's optimal allocation to contest 1. The dashed line illustrates competitor i's best response correspondence when $\alpha = 1$. The solid line indicates competitor i's best response correspondence when $\alpha = 8$. The dotted line indicates the equilibrium allocation to contest 1. If competitor j selects a nonequilibrium allocation, then competitor i's best response is always closer to equilibrium under $\alpha = 1$ than $\alpha = 8$. Deviations from equilibrium by one competitor incentivize larger deviations from equilibrium by the other competitor when $\alpha = 8$ than when $\alpha = 1$.

Figure 2 illustrates the equilibrium payoff function for the resource allocation game with $c = 1$, $\beta = 28$, and $v_1 = x_{j1} = 0.8$. The horizontal axis indicates competitor i's investment in contest 1 and the vertical axis indicates competitor i's payoff. The dashed line illustrates competitor i's payoff function when $\alpha = 1$. The solid line illustrates competitor i's payoff function when $\alpha = 8$. The dotted line indicates competitor j's allocation to contest 1. In both cases, the equilibrium allocation is $x_{i1} = 0.8$ and the equilibrium payoff is $\pi_i(x) = 0.5$. If a competitor expects their opponent to allocate their resources in accordance with equilibrium predictions then they have a stronger incentive to closely approximate equilibrium predictions when $\alpha = 8$ than when $\alpha = 1$.

A Nash equilibrium is said to be evolutionarily stable if deviations from equilibrium by a sufficiently small fraction of the population always give the equilibrium strategy a higher expected payoff than than the deviating strategy (Taylor and Jonker, 1978). More formally, a symmetric Nash equilibrium (σ^*, σ^*) is said to be evolutionarily stable if, for any nonequilibrium mixed strategy $\sigma \neq \sigma^*$ and any sufficiently small $\varepsilon > 0$, $\pi_1(\sigma, \bar{\sigma}) < \pi_1(\sigma^*, \bar{\sigma})$ where $\bar{\sigma} = \varepsilon\sigma + (1 - \varepsilon)\sigma^*$ denotes a mixed strategy that involves utilizing the the nonequilibrium strategy σ with probability ε and utilizing the equilibrium strategy σ^* with probability $1 - \varepsilon$. Intuitively, such equilibria are stable because rare deviations from equilibrium never incentivize equilibrium players to adopt the deviating strategy. As shown in the proof of Theorem 1, the objective function π_i is strictly quasiconcave in x_i, so the Nash equilibrium x^* is strict and the equilibrium strategy is evolutionarily stable.

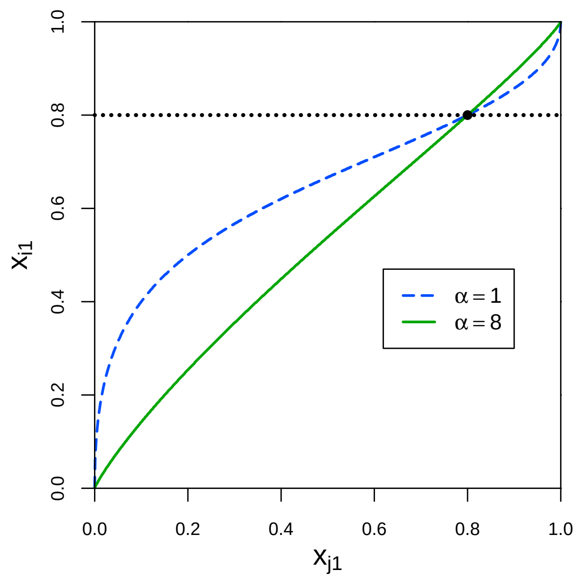

Figure 1: Best responses for $c = 1$, $\beta = 28$, and $v_1 = 0.8$

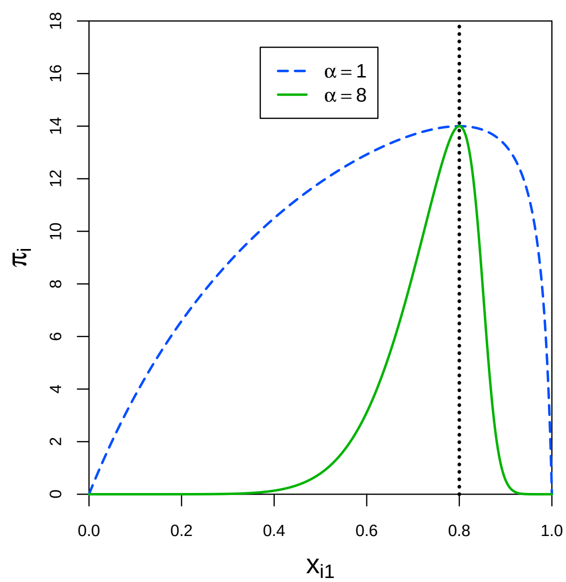

Figure 2: Payoff functions for $c = 1$, $\beta = 28$, and $v_1 = x_{j1} = 0.8$

A symmetric two player game with a one dimensional strategy space is said to be supermodular if the marginal payoff to one player from increasing their strategy is increasing in the other player's strategy. The resource allocation game described above is symmetric and the strategy space is a one dimensional unit simplex. Differentiating player i's marginal benefit from increasing their allocation to contest 1 with respect to competitor j's allocation to contest 1 yields

$$\frac{\partial^2 \pi_i}{\partial x_{i1} \partial x_{j1}} = \alpha\beta\pi_i\left(x\right)^2 \left[\frac{v_1}{x_{i1}^2} + \frac{v_2}{x_{i2}^2}\right] \tag{5}$$

Since this expression is strictly positive for all $x_i \in \mathbb{R}^2_{++}$, the marginal payoff to player i from reallocating additional resources to contest 1 is increasing in competitor j's allocation to contest 1, so the resource allocation game is supermodular.

3 Experimental Design

Each session implemented one of the four treatment conditions described in Table 1. A total of 8 experimental sessions were conducted, 2 for each of the 4 treatment conditions. Each of the 8 sessions was conducted with 20 subjects for a total of 160 experimental subjects. At the beginning of each session, subjects were randomly matched into pairs which remained fixed over the entire session. Each experimental session consisted of 100 periods.

During each period, each subject allocated 100 tokens between two contests. The share of a given prize awarded to a given subject was proportional to a power function of their investment in the corresponding contest. The success function (1) was less sensitive to investment levels in the low responsiveness treatment ($\alpha = 1$) and more sensitive to investment levels in the high responsiveness treatment ($\alpha = 8$).

Valuation treatments were constructed symmetrically to control for the possibility of labeling or ordering biases between contests. In every treatment, one of the contests had a factor value of $v_i = 0.8$ and the other contest had a factor value of $v_j = 0.2$. In the first valuation treatment, the first contest had the

	Low Responsiveness	High Responsiveness
First Valuation	$\alpha = 1,\ v_1 = 0.8$	$\alpha = 8,\ v_1 = 0.8$
Second Valuation	$\alpha = 1,\ v_1 = 0.2$	$\alpha = 8,\ v_1 = 0.2$

Table 1: Experimental Design

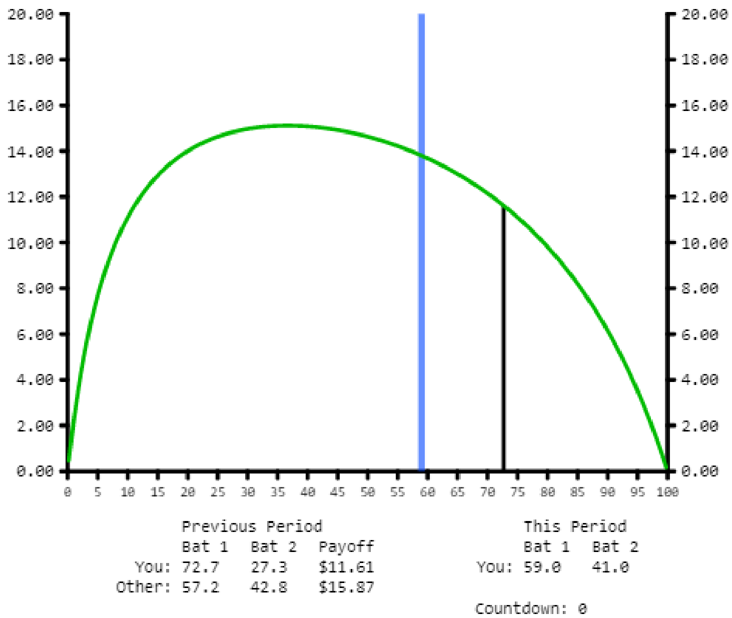

Figure 3: Experimental Interface

higher factor value, so the valuation vector was given by $v = (0.8, 0.2)$. In the second valuation treatment, the second contest had the higher factor value, so the valuation vector was given by $v = (0.2, 0.8)$.

At the end of each period, a subject's payoff was given by a unit elasticity aggregator (2) of their prize shares with scale factor $\beta = 28$. At the end of each session, subjects received their average payoff over all periods plus a $7 participation bonus. Average earnings were $19.92 per subject.

At the end of each period, subjects received feedback about their allocations and payoffs. Figure 3 depicts the experimental interface. The horizontal axis indicates the number of tokens the subject invested in contest 1 and the vertical axis indicates the subject's payoff. The horizontal position of the black line indicates the number of tokens allocated to contest 1 in the previous period. The height of the black line indicates their payoff. The horizontal position of the blue line indicates the number of tokens allocated to contest 1 in the current period. Additional information about allocations and payoffs is provided below the graph.

4 Hypotheses

In equilibrium, resources are allocated to each contest in proportion to it's factor value. Every treatment has a unique Nash equilibrium where 80% of resources are invested in the high value contest and 20% resources are invested in the low value contest.

Hypothesis 1. *More resources will be allocated to the high value contest than the low value contest.*

Theorem 2 states that best responses to nonequilibrium resource allocations are closer to equilibrium predictions when contest success functions are less responsive to investment levels. Accordingly, we may expect the observed resource allocations to approximate equilibrium predictions more closely when success functions are less responsive to resource investment levels.

Hypothesis 2. *Resource allocations will be closer to equilibrium predictions in settings with less responsive success functions.*

Theorem 3 states that competitors have stronger incentives to select equilibrium allocations in response to equilibrium allocations when success functions are more responsive to resource allocation levels if others select equilibrium allocations. Accordingly, we may expect the observed resource allocations to approximate equilibrium predictions more closely when success functions are more responsive to resource investment levels.

Hypothesis 3. *Resource allocations will be closer to equilibrium predictions in settings with more responsive success functions.*

5 Results

Figure 4 illustrates the empirical cumulative distribution function for the observed resource allocations. The horizontal axis indicates the percent of a subject's resources invested in the high value contest. The vertical axis indicates the percentage of observed allocations at or below the given level. The solid line depicts the empirical cumulative distribution function under the low responsiveness treatment where $\alpha = 1$. The dashed line indicates the empirical cumulative distribution function under the high responsiveness treatment where $\alpha = 8$. The dotted vertical line indicates the predicted share of resources allocated to the high value contest in equilibrium.

In equilibrium, agents should allocate 80% of their resources to the high value contest and 20% of their resources to the low value contest. In the experiment, subjects allocated 73.4% of their resources to the high value contest and 26.6% of their resources to the low value contest on average. Consistent with Hypothesis 1, both a nonparametric Wilcoxon signed-rank test and a t-test find this difference to be significant at the 1% level. These hypothesis tests are reported in Table 2. The average allocation selected by a fixed matching pair over the entire experimental session is treated as a single observation. There were 4 sessions per valuation treatment and 10 fixed matching pairs per session, yielding a total of 40 observations.

Result 1. *More resources were allocated to the high value contest than the low value contest.*

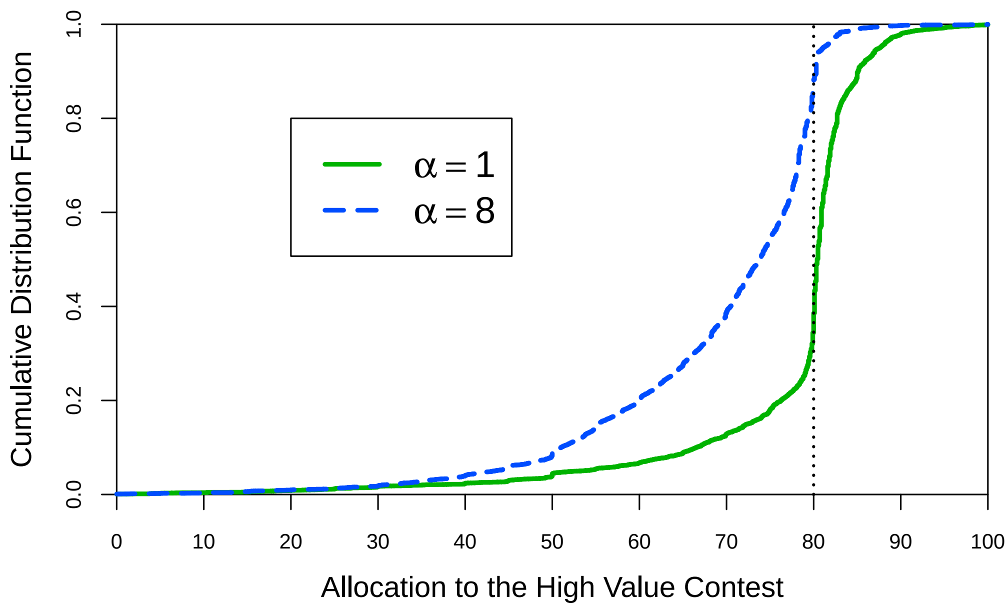

Figure 4: CDF of Investment in the High Value Contest

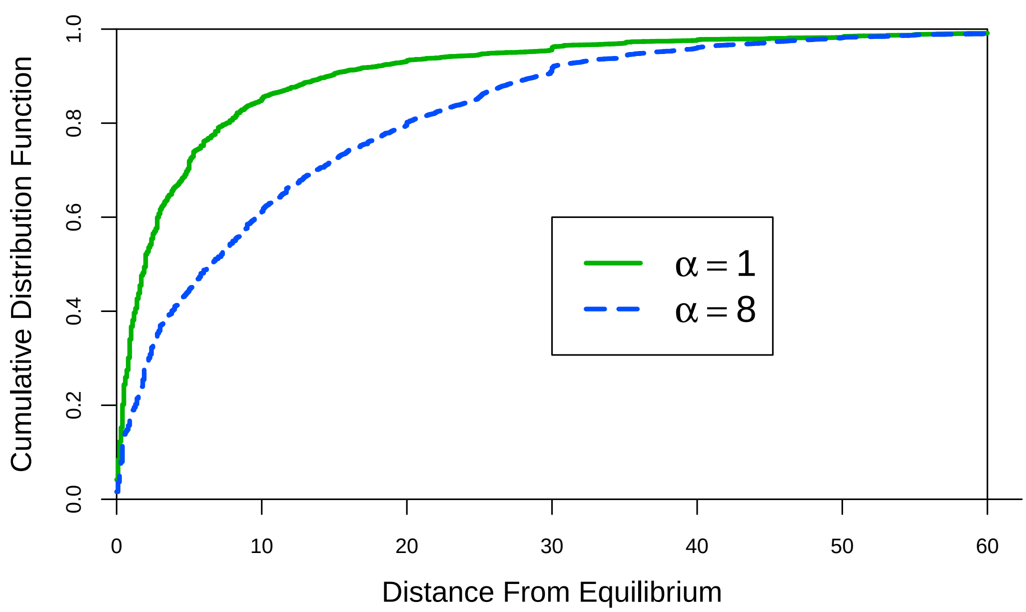

Figure 5: CDF of Distance from Equilibrium

			p-value	
	$\alpha = 1$	$\alpha = 8$	rank-sum	t-test
Distance from Equilibrium	5.73	11.21	< 0.0001	< 0.0001

			p-value	
	Low Value	High Value	signed-rank	t-test
Average Allocation	26.6	73.4	< 0.0001	< 0.0001

Table 2: Hypothesis Tests

Figure 5 illustrates the empirical cumulative distribution function for distance from equilibrium under each responsiveness treatment. Distance from equilibrium is defined as the absolute difference between observed resource allocations and equilibrium resource allocations. The horizontal axis indicates distance from equilibrium. The vertical axis indicates the percent of distances at or below a given level. The solid line is the empirical cumulative distribution function for the low responsiveness treatment where $\alpha = 1$. The dashed line is the empirical cumulative distribution function for the high responsiveness treatment where $\alpha = 8$.

The average deviation from equilibrium in the low responsiveness treatment was 5.73 while the average deviation from equilibrium in the high responsiveness treatment was 11.21. As shown in Figure 5, the distribution of distances from equilibrium in the high responsiveness treatment first-order stochastically dominates the distribution of distances from equilibrium in the low responsiveness treatment. Consistent with consistent with Hypothesis 2, observed resource allocations were significantly closer to equilibrium predictions under less responsive contest success functions.

Result 2. *Resource allocations were significantly closer to equilibrium predictions in the low responsiveness treatment than the high responsiveness treatment.*

Both a t-test and a Wilcoxon rank-sum test find this difference to be statistically significant at the 1% level. In both of these hypothesis tests, the average allocation selected by a fixed matching pair over an entire experimental session is treated as a single observation, yielding a total of 40 observations. These hypothesis tests are reported in Table 2.

If a competitor expects their opponent to follow equilibrium predictions, then a more responsive success function strengthens their incentive to closely approximate the equilibrium strategy. However, if a competitor expects their opponent to employ a nonequilibrium strategy, then a more responsive success function brings their best response further away from equilibrium, which may explain why subjects in this experiment consistently exhibited larger deviations from equilibrium under more responsive success functions. These results suggest that nonequilibrium incentives had a significant impact on the reliability of equilibrium predictions.

6 Conclusions

This paper experimentally investigates conflicts where competitors allocate resources to compete for shares of complementary factors. Factor shares are proportional to a power function of investment levels and serve as complementary inputs to objective functions. Competitors have stronger incentives to follow equilibrium predictions in response to an equilibrium strategy when contest success functions are more responsive to resource allocations. Conversely, best responses to nonequilibrium strategies exhibit larger deviations from from equilibrium under more responsive contest success functions.

The experimental design varies the responsiveness of the success function across treatment conditions. Both treatments have identical equilibrium predictions, but observed allocations exhibited significantly larger deviations from equilibrium under more responsive success functions. Observed behavior may have approximated equilibrium predictions more closely when success functions were less responsive to investment levels because best responses to nonequilibrium allocations are closer to equilibrium predictions under less responsive success functions. These results suggest that nonequilibrium incentives can have a significant impact on the reliability of equilibrium predictions in conflicts over complementary factors.

The present study considers a particular class of conflicts where competitors allocate resources to compete for shares of complementary factors. The experimental design of the present study varies the responsiveness of the success function across treatment conditions, but it does not vary the number of competitors or the level of complementarity between factors. Further study is needed to investigate how these features of the allocation game influence resource behavior. Additional research is needed to better understand how nonequilibrium incentives influence the reliability of equilibrium predictions in other strategic settings.

A Proofs

Proof of Theorem 1. Suppose competitor i allocates zero resources to contest k such that $x_{ik} = 0$. Let $\hat{x}_i$ such that $\hat{x}_{ik} = \varepsilon \in (0, 1)$ and $\hat{x}_{ib} = 1 - \varepsilon$. If $x_{ik} = 0$ and $x_{jk} \neq 0$ then $\pi_i(x) = 0 < \pi_i(\hat{x}_i, x_j)$. If $x_{ik} = x_{jk} = 0$ then taking the limit as $\varepsilon \to 0$ obtains (6).

$$\lim_{\varepsilon \to 0} \pi_i(\hat{x}_i, x_j) = \frac{\beta}{v_k + 2v_b} > \frac{\beta}{2} = \pi_i(x) \tag{6}$$

Hence $x_{ik} > 0$ in every Nash equilibrium. Differentiating $g_i(x) = -\frac{\beta}{\pi_i(x)}$ with respect to x_{ik} yields (7).

$$\frac{\partial g_i}{\partial x_{ik}} = \frac{\alpha v_k [1 - y_{ik}(x)]}{y_{ik}(x)\, x_{ik}} \tag{7}$$

By (1), we have $\frac{\partial y_{ik}(x)}{\partial x_{ik}} > 0 = \frac{\partial y_{ik}(x)}{\partial x_{ib}}$ for $b \neq k$, so $\frac{\partial^2 g_{ik}(x)}{\partial x_{ik}^2} < 0 = \frac{\partial^2 g_{ik}(x)}{\partial x_{ib}^2}$ by (7). Hence g_i is strictly concave in x_i, so π_i is strictly quasiconcave in x_i. Since π_i is strictly quasiconcave in x_i, the first order conditions (8) are necessary and sufficient for equilibrium.

$$\frac{v_k [1 - y_{ik}(x)]}{y_{ik}(x)\, x_{ik}} = \frac{v_b [1 - y_{ib}(x)]}{y_{ib}(x)\, x_{ib}} \tag{8}$$

Solving (8) for $\frac{x_{ik}}{x_{ib}}$ obtains (9).

$$\frac{v_k y_{jk}(x) y_{ib}(x)}{v_b y_{jb}(x) y_{ik}(x)} = \frac{x_{ik}}{x_{ib}} \tag{9}$$

Since the left hand side of (9) is identical for $i = 1$ and $i = 2$, we have (10).

$$\frac{x_{ik}}{x_{ib}} = \frac{x_{jk}}{x_{jb}} \tag{10}$$

Hence $y_{ik}(x) = y_{ib}(x)$ by (1). Substituting this into equation (9) yields (11).

$$\frac{v_k}{v_b} = \frac{x_{ik}}{x_{ib}} \tag{11}$$

Since $x_{i1} + x_{i2} = v_1 + v_2 = 1$ we have $x_{ik} = v_k$. □

Proof of Theorem 2. If x_i is a best response to x_j then by (9)

$$\frac{x_{ik}}{x_{ib}} = \frac{v_k y_{ib}(x) y_{jk}(x)}{v_b y_{ik}(x) y_{jb}(x)} \tag{12}$$

Substituting (1) into (12) yields (13).

$$\frac{x_{ik}}{x_{ib}} = \frac{v_k x_{ib}^{\alpha} x_{jk}^{\alpha}}{v_b x_{ik}^{\alpha} x_{jb}^{\alpha}} \tag{13}$$

Rearranging (13) produces (14).

$$\frac{x_{ik}}{x_{ib}} = \left(\frac{v_k}{v_b}\right)^{\frac{1}{1+\alpha}} \left(\frac{x_{jk}}{x_{jb}}\right)^{\frac{\alpha}{1+\alpha}} \tag{14}$$

Substituting $x_{ib} = 1 - x_{ik}$ and $v_b = 1 - v_k$ into (14) obtains (15).

$$\frac{x_{ik}}{1 - x_{ik}} = \left(\frac{v_k}{1 - v_k}\right)^{\frac{1}{1+\alpha}} \left(\frac{x_{jk}}{1 - x_{jk}}\right)^{\frac{\alpha}{1+\alpha}} \tag{15}$$

Solving (15) for x_{ik} yields (16).

$$x_{ik} = \frac{\left(\frac{v_k}{1-v_k}\right)^{\frac{1}{1+\alpha}} \left(\frac{x_{jk}}{1-x_{jk}}\right)^{\frac{\alpha}{1+\alpha}}}{1 + \left(\frac{v_k}{1-v_k}\right)^{\frac{1}{1+\alpha}} \left(\frac{x_{jk}}{1-x_{jk}}\right)^{\frac{\alpha}{1+\alpha}}} \tag{16}$$

Differentiating (16) with respect to x_{jk} obtains (17).

$$\frac{\partial x_{ik}}{\partial x_{jk}} = \frac{\left(\frac{\alpha}{1+\alpha}\right) \left(\frac{v_k}{1-v_k}\right)^{\frac{1}{1+\alpha}} \left(\frac{x_{jk}}{1-x_{jk}}\right)^{\frac{\alpha}{1+\alpha}}}{x_{jk}\left(1-x_{jk}\right) \left[1 + \left(\frac{v_k}{1-v_k}\right)^{\frac{1}{1+\alpha}} \left(\frac{x_{jk}}{1-x_{jk}}\right)^{\frac{\alpha}{1+\alpha}}\right]^2} > 0 \tag{17}$$

Hence competitor i's optimal allocation to contest k is strictly increasing in competitor j's allocation to contest k, so competitor i's best response satisfies (18) by Theorem 1.

$$\operatorname{sgn}\left(x_{ik} - v_k\right) = \operatorname{sgn}\left(x_{jk} - v_k\right) \tag{18}$$

Differentiating (16) with respect to α obtains (19).

$$\frac{\partial x_{ik}}{\partial \alpha} = \frac{\left(\frac{v_k}{v_b}\right)^{\frac{1}{1+\alpha}} \left(\frac{x_{jk}}{1-x_{jk}}\right)^{\frac{\alpha}{1+\alpha}} \left[\log\left(\frac{x_{jk}}{1-x_{jk}}\right) - \log\left(\frac{v_k}{1-v_k}\right)\right]}{(1+\alpha)^2 \left[1 + \left(\frac{v_k}{1-v_k}\right)^{\frac{1}{1+\alpha}} \left(\frac{x_{jk}}{1-x_{jk}}\right)^{\frac{\alpha}{1+\alpha}}\right]} \tag{19}$$

Combining (18) and (19) yields (20).

$$\operatorname{sgn}\left(\frac{\partial x_{ik}}{\partial \alpha}\right) = \operatorname{sgn}\left(x_{jk} - v_k\right) = \operatorname{sgn}\left(x_{ik} - v_k\right) \tag{20}$$

By (20), differentiating $|x_{ik} - v_k|$ with respect to α produces (21).

$$\frac{\partial\left|x_{ik} - v_k\right|}{\partial \alpha} = \frac{\left(x_{ik} - v_k\right)}{\left|x_{ik} - v_k\right|} \frac{\partial x_{ik}}{\partial \alpha} > 0 \tag{21}$$

□

Proof of Theorem 3. Let $g_i\left(x_{ik}\right) = -\frac{\beta}{\pi_i(x_i,v)}$. By (1) and (2) and we have (22).

$$g_i\left(x_{ik}\right) = -v_k\left(\frac{x_{ik}^{\alpha}+v_k^{\alpha}}{x_{ik}^{\alpha}}\right) - v_\ell\left(\frac{x_{i\ell}^{\alpha}+v_\ell^{\alpha}}{x_{i\ell}^{\alpha}}\right) \tag{22}$$

Substituting $x_{i\ell} = 1 - x_{ik}$ and $v_\ell = 1 - v_k$ into (22) produces

$$g_i\left(x_{ik}\right) = -1 - \frac{v_k^{\alpha+1}}{x_{ik}^{\alpha}} - \frac{\left(1-v_k\right)^{\alpha+1}}{\left(1-x_{ik}\right)^{\alpha}} \tag{23}$$

Differentiating (23) with respect to α yields (24).

$$\frac{\partial g_i}{\partial \alpha} = \frac{v_k^{\alpha+1}}{x_{ik}^{\alpha}}\log\left(\frac{x_{ik}}{v_k}\right) + \frac{\left(1-v_k\right)^{\alpha+1}}{\left(1-x_{ik}\right)^{\alpha}}\log\left(\frac{1-x_{ik}}{1-v_k}\right) \tag{24}$$

Differentiating (24) with respect to x_{ik} obtains (25)

$$\begin{aligned}\frac{\partial^2 g_i}{\partial\alpha\partial x_{ik}} = {} & \left(\frac{v_k}{x_{ik}}\right)^{\alpha+1} - \left(\frac{1-v_k}{1-x_{ik}}\right)^{\alpha+1} + \\ & \alpha\left(\frac{1-v_k}{1-x_{ik}}\right)^{\alpha+1}\log\left(\frac{1-x_{ik}}{1-v_k}\right) - \alpha\left(\frac{v_k}{x_{ik}}\right)^{\alpha+1}\log\left(\frac{x_{ik}}{v_k}\right)\end{aligned} \tag{25}$$

If $x_{ik} < v_k$ then $\frac{\partial^2 g_i}{\partial\alpha\partial x_{ik}} > 0$ by (25). Conversely, if $x_{ik} > v_k$ then $\frac{\partial^2 g_i}{\partial\alpha\partial x_{ik}} < 0$ by (25). Hence $\frac{\partial g_i}{\partial \alpha}$ is uniquely maximized at $x_{ik} = v_k$ where $\frac{\partial g_i}{\partial \alpha} = 0$ by (24). Hence $\frac{\partial g_i}{\partial \alpha} < 0$ for $v_k \neq x_{ik} \neq 0$. Since $\pi_i\left(x_i, v\right)$ is strictly increasing in $g_i\left(x_{ik}\right)$, $\pi_i\left(v, v\right) = \frac{1}{2}\beta$, and $\frac{\partial g_i}{\partial \alpha} < 0$ for $v_k \neq x_{ik} \neq 0$ we have (26).

$$\frac{\partial\left|\pi_i\left(v,v\right) - \pi_i\left(x_i,v\right)\right|}{\partial\alpha} > 0 \tag{26}$$

□

References

Kyung Hwan Baik, Subhasish M Chowdhury, and Abhijit Ramalingam. The effects of conflict budget on the intensity of conflict: An experimental investigation. *Experimental Economics*, 23:240–258, 2020.

Emile Borel. La théorie du jeu et les équations intégralesa noyau symétrique. *Comptes rendus de l'Académie des Sciences*, 173(1304-1308):58, 1921. English translation by Leonard J. Savage in Econometrica 21(1):97-100, 1953.

Timothy N Cason, Daniel Friedman, and Ed Hopkins. Cycles and instability in a rock–paper–scissors population game: A continuous time experiment. *Review of Economic Studies*, 81(1):112–136, 2014.

Timothy N Cason, William A Masters, and Roman M Sheremeta. Winner-take-all and proportional-prize contests: theory and experimental results. *Journal of Economic Behavior & Organization*, 175:314–327, 2020.

Yan Chen and Robert Gazzale. When does learning in games generate convergence to nash equilibria? the role of supermodularity in an experimental setting. *American Economic Review*, 94(5):1505–1535, 2004.

Subhasish M Chowdhury, Dan Kovenock, David Rojo Arjona, and Nathaniel T Wilcox. Focality and asymmetry in multi-battle contests. *The Economic Journal*, 131(636):1593–1619, 2021.

John Duffy and Alexander Matros. Stochastic asymmetric blotto games: An experimental study. *Journal of Economic Behavior & Organization*, 139: 88–105, 2017.

Aniol Llorente-Saguer, Roman M Sheremeta, and Nora Szech. Designing contests between heterogeneous contestants: An experimental study of tie-breaks and bid-caps in all-pay auctions. *European Economic Review*, 154: 104327, 2023.

Bruce R Pirnie, Alan Vick, Adam Grissom, Karl P Mueller, and David T Orletsky. Beyond close air support. forging a new air-ground partnership. Technical report, Rand Corporation, Santa Monica, CA, 2005.

Jean-Charles Rochet and Jean Tirole. Platform competition in two-sided markets. *Journal of the european economic association*, 1(4):990–1029, 2003.

Daniel Stephenson. Assignment feedback in school choice mechanisms. *Experimental Economics*, 25(5):1467–1491, 2022.

Daniel Stephenson. Multi-battle contests over complementary battlefields. *Review of Economic Design*, pages 1–16, 2024.

Peter D Taylor and Leo B Jonker. Evolutionarily stable strategies and game dynamics. *Mathematical biosciences*, 40(1-2):145–156, 1978.

G. Tullock. Efficient rent-seeking. In J.M. Buchanan, R.D. Tollison, and G. Tullock, editors, *Toward a Theory of the Rent Seeking Society*, pages 97–112. Texas A & M University Press, 1980.